NATIONAL MATH + SCIENCE INITIATIVE

AP Calculus

Limits, Continuity, and Differentiability

Presenter Notes

2017-2018 EDITION

Student Study Session - Presenter Notes

Thank you for agreeing to present at one of NMSI's Saturday Study Sessions. We are grateful you are sharing your time and expertise with our students. Saturday mornings can be a "tough sell" for students, so we encourage you to incorporate strategies and techniques to encourage student movement and engagement. Suggestions for different presentation options are included in this document. If you have any questions about the content or about presentation strategies, please contact Mathematics Director Charla Holzbog at cholzbog@nms.org or AP Calculus Content Specialist Karen Miksch at kmiksch@nms.org.

The material provided contains many released AP multiple choice and free response questions as well as some AP-like questions that we have created. The goal for the session is to let the students experience a variety of both types of questions to gain insight on how the topic will be presented on the AP exam. It is also beneficial for the students to hear a voice other than their teacher in order to help clarify their understanding of the concepts.

Suggestions for presenting:
The vast majority of the study sessions are on Saturday and students and teachers are coming to be WOWed! We want activities to engage the students as well as prepare them for the AP Exam. The following presenter notes include pacing suggestions (you only have 50 minutes!), solutions, and recommended engagement strategies.

Suggestions on how to prepare:
- The notes/summaries on the last page(s) are for reference. We want the students' time during the session to be focused on the questions as much as possible and not taking or reading the notes. As the questions are presented during the session, you may wish to refer the students back to those pages as needed. It is not our intent for the sessions to begin with a lecture over these pages.
- As you prepare, work through the questions in the packet noting the level of difficulty and topic or skill required for the questions.
- Design a plan for what questions you would like to cover with the group depending on their level of expertise. Some groups will be ready for the tougher questions while other groups will need more guidance and practice on the easier ones. Create an easy, medium, and hard listing of the questions prior to the session. This will allow you to adjust on the fly as you get to know the groups. In most instances, there will **not** be enough time to cover all the questions in the packet. Use your judgement on the amount of questions to cover based on the students' interactions. Remember to include both multiple choice and free response type questions. Discussions on test taking strategies and scoring of the free response questions are always great to include during the day.
- The concepts should have been previously taught; however, be prepared to "teach" the topic if you find out the students have not covered the concept prior in class. In sessions where multiple schools come together, you might have a mixture of students with and without prior knowledge on the topic. You will have to use your best judgement in this situation.
- Consider working through some free response questions before the multiple choice questions, or flipping back and forth between the two types of questions. Sometimes, if free response questions are saved for the last part of the session, it is possible students only get practice with one or two of them and most students need additional practice with free response questions.

Limits, Continuity, and Differentiability Student Session-Presenter Notes

This session includes a reference sheet at the back of the packet since for most students it has been some time since they have studied limits; however we suggest that teachers continuously review limits throughout the year. Therefore we suggest that the presenter does not spend time going over the reference sheet, but may point it out to students who are struggling and also as a general announcement that it is available for students to refer to if needed. It is not meant to be a complete study of all limit techniques and properties but rather a reference for some of the types of limit questions that appear most frequently on the AP exam.

I. 10 Minute Group Activity

- Arrange the students randomly into groups of 3 or 4. Ask students to work together on questions 1-5 while you actively walk around to monitor their progress and assist with any questions. Use this time to gauge the level of knowledge of the group. Notice in the solutions guide the questions are categorized as 3, 4, or 5 indicating a typical question of the difficulty level (DL) for a student earning these qualifying scores on the AP exam.

- After a few minutes check answers with students and clarify any misunderstandings. Below are suggested questions that are similar to each of the questions 1 through 5 that you may assign to the group as additional practice during the next 20 minutes along with some of the other types of questions.

II. 20 minutes Multiple Choice Additional Practice-*your choice of questions depending on student struggles during the first ten minutes—some suggested questions are listed below.*

- Question 1: #12 (review how to use the ratio of the leading coefficients here to help find the limit quickly, but discuss L'Hospital's rule as another option. Also note the connection of this limit to the horizontal asymptote at $y = 4.$)
- Question 2: #18 (Alternative limit definition of the derivative or L'Hospital's Rule-be aware that some students may not have learned L'Hospital's rule before this study session so might need to teach)
- Question 3: #13,14
- Question 4: #17
- Question 5: #3,7—since L'Hospital's Rule is returning for the 2017 AB exam, be sure to spend some time with these questions.
- Choose other questions that also go over rules for determining the limit as x approaches infinity to determine horizontal asymptotes such as #10 and #11.
- Note on the next page of the trainer notes we have listed questions to go with the reference guide to help guide in your selection of questions depending on the needs of your group of students.

III. 20 minutes Free Response

- Model how to do the free response question involving continuity by working through #20 with the students.
- Students try #21 then grade with the rubric.
- Have students grade selected students samples using the rubric as a guide.
- Debrief last five minutes—what have the students found to be valuable about grading student samples?

Limits, Continuity, and Differentiability Reference Page With Associated Question Numbers

Existence of a Limit at a Point (#5, 9, 13, 14, 17)

A function $f(x)$ has a limit L as x approaches c if and only if the left-hand and right-hand limits at c exist and are equal.

1. $\lim_{x \to c^-} f(x)$ exists
2. $\lim_{x \to c^+} f(x)$ exists
3. $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$ $\qquad \therefore \lim_{x \to c} f(x) = L$

Continuity (#4,13,15,16,20,21)

A function is continuous on an interval if it is continuous at every point of the interval. Intuitively, a function is continuous if its graph can be drawn without ever needing to pick up the pencil. This means that the graph of $y = f(x)$ has no "holes", no "jumps" and no vertical asymptotes at $x = a$. When answering free response questions on the AP exam, the formal definition of continuity is required. To earn all of the points on the free response question scoring rubric, all three of the following criteria need to be met, with work shown:

A function is _continuous_ at a point $x = a$ if and only if:

1. $f(a)$ exists
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$ (i.e., the limit equals the function value)

Limit Definition of a Derivative (#2)

The derivative of a function $f(x)$ with respect to x is the function $f'(x)$ whose value at x is

$f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$, provided the limit exists.

Alternative Form for Definition of a Derivative (#18)

The derivative of a function $f(x)$ at $x = a$ is $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$, provided the limit exists.

Continuity and Differentiability (#4, 13)
Differentiability implies continuity (but not necessarily vice versa) If a function is differentiable at a point (at every point on an interval), then it is continuous at that point (on that interval). The converse is not always true: continuous functions may not be differentiable. It is possible for a function to be continuous at a specific value for a but not differentiable at a.

L'Hospital's Rule (returns on 2017 AB exam) (#3,6,7,18)
Given that f and g are differentiable functions on an open interval (a, b) containing c (except possibly at c itself), assume that $g'(x) \neq 0$ for all x in the interval (except possibly at c itself).
If $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)}$ produces the indeterminate form $\dfrac{0}{0}$, then $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to c} \dfrac{f'(x)}{g'(x)}$
provided the limit on the right exists (or is infinite). This result also applies when the limit produces any one of the indeterminate forms $\dfrac{\infty}{\infty}, \dfrac{-\infty}{\infty}, \dfrac{\infty}{-\infty},$ or $\dfrac{-\infty}{-\infty}$.

Note the work required on the free response rubric for 2016 BC4c:

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Question 4

Consider the differential equation $\dfrac{dy}{dx} = x^2 - \dfrac{1}{2}y$.

(c) Let $y = g(x)$ be the particular solution to the given differential equation with $g(-1) = 2$. Find $\lim\limits_{x \to -1}\left(\dfrac{g(x) - 2}{3(x+1)^2}\right)$. Show the work that leads to your answer.

(c) $\lim\limits_{x \to -1}(g(x) - 2) = 0$ and $\lim\limits_{x \to -1} 3(x+1)^2 = 0$ Using L'Hospital's Rule, $\lim\limits_{x \to -1}\left(\dfrac{g(x) - 2}{3(x+1)^2}\right) = \lim\limits_{x \to -1}\left(\dfrac{g'(x)}{6(x+1)}\right)$ $\lim\limits_{x \to -1} g'(x) = 0$ and $\lim\limits_{x \to -1} 6(x+1) = 0$ Using L'Hospital's Rule, $\lim\limits_{x \to -1}\left(\dfrac{g'(x)}{6(x+1)}\right) = \lim\limits_{x \to -1}\left(\dfrac{g''(x)}{6}\right) = \dfrac{-2}{6} = -\dfrac{1}{3}$	$3 : \begin{cases} 2 : \text{L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$

Limits, Continuity, and Differentiability Solutions

We have intentionally included more material than can be covered in most Student Study Sessions to account for groups that are able to answer the questions at a faster rate. Use our presenter notes and your own judgment, based on the group of students, to determine the order and selection of questions to work in the session. Be sure to include a variety of types of questions (multiple choice, free response, calculator, and non-calculator) in the time allotted. Notice in the solutions guide the questions are categorized as 3, 4, or 5 indicating a typical question of the difficulty level (DL) for a student earning these qualifying scores on the AP exam.

Multiple Choice

1. D (1985 AB5) DL: 3
   Since the limit is taken as $n \to \infty$ and the exponents in the numerator and denominator are equal, use the ratio of the leading coefficients to find that the limit is $\frac{4n^2}{n^2} = 4$.
2. B (1988 AB29) DL: 4
   This limit represents the derivative of the function, $f(x) = \tan 3x$. Using the chain rule, $f'(x) = 3\sec^2(3x)$
3. B (AB Sample Question #2 from AP Calculus Course and Exam Description) DL: 4
   $\lim_{x\to 0}(7x - \sin x) = 0$ and $\lim_{x\to 0}(x^2 + \sin(3x)) = 0$,
   therefore by L'Hospital's Rule, $\lim_{x\to 0}\frac{7x-\sin x}{x^2+\sin(3x)} = \lim_{x\to 0}\frac{7-\cos x}{2x+3\cos(3x)} = \frac{7-\cos 0}{0+3} = \frac{7-1}{3} = 2$
4. E (1988 AB27) DL: 4
   $f(3) = 6(3) - 9 = 9$
   $\lim_{x\to 3^-} x^2 = \lim_{x\to 3^+} 6x - 9 = 9$
   Since $f(3) = \lim_{x\to 3} f(x)$ the function is continuous at $x = 3$
   $f'(x) = \begin{cases} 2x, & x < 3 \\ 6, & x > 3 \end{cases}$ and $\lim_{x\to 3^-} 2x = \lim_{x\to 3^+} 6 = 6$
   Since the left and right limits of the derivative of the function are equivalent from either side of 3, the function is differentiable at $x = 3$.
5. C (2008 AB77) DL: 3
   $\lim_{x\to 2^-} f(x)$ and $\lim_{x\to 2^+} f(x)$ exist; however, since they are not equivalent, the $\lim_{x\to 2} f(x)$ does not exist.
6. C (1993 BC2 appropriate for AB) DL: 3
   $\lim_{x\to 0}\frac{2x^2+1-1}{x^2} = 2$
7. B (1988 AB23) DL: 4
   Since $f'(x) = \cos x$, $f(x) = \sin x$. Also $f(0) = 0$, $g(0) = 0$, and $g'(x) = 1$, hence $\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{\sin x}{x} = 1$.
   Alternatively, by L'Hospital's rule, $\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)} = \frac{f'(0)}{g'(0)} = \frac{\cos 0}{1} = 1$.

8. B (AP-like) DL: 5

$\lim\limits_{x\to 2^-} f(x)=0$; $\lim\limits_{x\to 2^+} f(x)=1$ therefore I is false; $\lim\limits_{x\to 2^-} f'(x)=4$; $\lim\limits_{x\to 2^+} f'(x)=4$ so II is true; III is false since f is not differentiable at $x=2$ since it is not continuous at $x=2$.

9. E (1985 AB41) DL: 4

Using the given limit, there is not enough information to establish that $f'(a)$ exists, nor that $f(x)$ is continuous or defined at $x=a$, nor that $f(a)=L$. For example, consider the function whose graph is the horizontal line $y=2$ with a hole at $x=a$. For this function $\lim\limits_{x\to a} f(x)=2$ and none of the given statements are true

10. E (2003 AB3) DL: 4

By definition of a horizontal asymptote, E is correct.

11. E (1993 AB35) DL: 5

Since $y=2$ is a horizontal asymptote, the ratio of the leading coefficients must be 2; therefore, $a=2$. Since there is a vertical asymptote at $x=-3$, set the denominator, $-3+c$, equal to 0, so $c=3$ then $a+c=2+3=5$.

12. B (2008 AB1) DL: 3

Multiplying in the numerator and denominator yields the equivalent limit:

$\lim\limits_{x\to\infty}\dfrac{-2x^2+7x-3}{x^2+2x-3}=-2$.

13. A (2008 AB6/BC6) DL: 4

Use the top piece of the piecewise function for the limit since the bottom piece gives the value of $f(2)$. Using factoring, $\lim\limits_{x\to 2}\dfrac{x^2-4}{x-2}=\lim\limits_{x\to 2}\dfrac{(x+2)(x-2)}{x-2}=\lim\limits_{x\to 2}x+2=4$. Since this limit does not equal $f(2)=1$, the function f is not continuous or differentiable at $x=2$.

14. A (AP-like) DL: 5

Differentiability implies continuity, so $2+2b=4a$ by substituting $x=2$ into the top and bottom piece of the function; Then using one-sided derivatives, $1=2a(2)$ when $x=2$.

Therefore, $a=\dfrac{1}{4}$ and $2+2b=4\left(\dfrac{1}{4}\right)$, so $b=-\dfrac{1}{2}$.

15. A (1993 AB5) DL: 4

Consider $f(a)=\lim\limits_{x\to a} f(x)$

Use factoring to simplify the function and then substitute for x:

$\lim\limits_{x\to 2}\dfrac{x^2-4}{x+2}=\lim\limits_{x\to 2}\dfrac{(x+2)(x-2)}{x+2}=x-2$; therefore, $f(-2)=-4$.

16. A (2003 AB13/BC13) DL: 3

The graph of f is continuous at $x=a$; however, since the graph has a sharp turn at $x=a$, the function is not differentiable at $x=a$.

17. D (2003 AB79) DL: 3
The one-sided limits as $x \to 4$ are equivalent for the graphs of f in I and II but not for III.

18. A (AP-like) DL: 4
$\lim_{x \to e} (\ln x - 1) = 0$ and $\lim_{x \to e} (x - e) = 0,$ therefore by L'Hospital's Rule,

$$\lim_{x \to e} \frac{\ln x - 1}{x - e} = \lim_{x \to e} \frac{\frac{1}{x}}{1} = \frac{1}{e}.$$

19. C (AB Sample Question #1 from AP Calculus Course and Exam Description) DL: 4

$$\lim_{x \to 1} f(g(x)) = f\left(\lim_{x \to 1} g(x)\right) = f(2) = 3$$

Free Response

20. 2011 AB6a

(a) $\lim_{x \to 0^-} (1 - 2\sin x) = 1$ $\lim_{x \to 0^+} e^{-4x} = 1$ $f(0) = 1$ So, $\lim_{x \to 0} f(x) = f(0)$. Therefore f is continuous at $x = 0$.	2: analysis

21. 2012 AB 4c

(c) $\lim_{x \to -3^-} g(x) = \lim_{x \to -3^-} f(x) = \lim_{x \to -3^-} \sqrt{25 - x^2 = 4}$ $\lim_{x \to -3^+} g(x) = \lim_{x \to -3^+} (x + 7) = 4$ Therefore, $\lim_{x \to -3} g(x) = 4$. $g(-3) = f(-3) = 4$ So, $\lim_{x \to -3} g(x) = g(-3)$.	$2 \begin{cases} 1: \text{ considers one-sided limits} \\ 1: \text{ answer with explanation} \end{cases}$

Student Samples Scoring Commentary for 4c

Sample 4A
The student earned both points.

Sample 4B
In part c the student earned 1 point for considering both one-sided limits. The student does not include $g(-3) = 4,$ so the explanation is incomplete.

Sample 4C
In part c the student does not use the definition of continuity so does not earn any points.

Sample 4D
The student earned 1 of the 2 points for considering both one-sided limits.

Sample 4E
The student's work is incorrect and does not earn any points.