NATIONAL MATH + SCIENCE INITIATIVE

AP Calculus

Analyzing a Function Based on its Derivatives

Presenter Notes

2017-2018 EDITION

Student Study Session - Presenter Notes

Thank you for agreeing to present at one of NMSI's Saturday Study Sessions. We are grateful you are sharing your time and expertise with our students. Saturday mornings can be a "tough sell" for students, so we encourage you to incorporate strategies and techniques to encourage student movement and engagement. Suggestions for different presentation options are included in this document. If you have any questions about the content or about presentation strategies, please contact Mathematics Director Charla Holzbog at cholzbog@nms.org or AP Calculus Content Specialist Karen Miksch at kmiksch@nms.org.

The material provided contains many released AP multiple choice and free response questions as well as some AP-like questions that we have created. The goal for the session is to let the students experience a variety of both types of questions to gain insight on how the topic will be presented on the AP exam. It is also beneficial for the students to hear a voice other than their teacher in order to help clarify their understanding of the concepts.

Suggestions for presenting:
The vast majority of the study sessions are on Saturday and students and teachers are coming to be WOWed! We want activities to engage the students as well as prepare them for the AP Exam. The following presenter notes include pacing suggestions (you only have 50 minutes!), solutions, and recommended engagement strategies.

Suggestions on how to prepare:

- The notes/summaries on the last page(s) are for reference. We want the students' time during the session to be focused on the questions as much as possible and not taking or reading the notes. As the questions are presented during the session, you may wish to refer the students back to those pages as needed. It is not our intent for the sessions to begin with a lecture over these pages.
- As you prepare, work through the questions in the packet noting the level of difficulty and topic or skill required for the questions.
- Design a plan for what questions you would like to cover with the group depending on their level of expertise. Some groups will be ready for the tougher questions while other groups will need more guidance and practice on the easier ones. Create an easy, medium, and hard listing of the questions prior to the session. This will allow you to adjust on the fly as you get to know the groups. In most instances, there will **not** be enough time to cover all the questions in the packet. Use your judgement on the amount of questions to cover based on the students' interactions. Remember to include both multiple choice and free response type questions. Discussions on test taking strategies and scoring of the free response questions are always great to include during the day.
- The concepts should have been previously taught; however, be prepared to "teach" the topic if you find out the students have not covered the concept prior in class. In sessions where multiple schools come together, you might have a mixture of students with and without prior knowledge on the topic. You will have to use your best judgement in this situation.
  Consider working through some free response questions before the multiple choice questions, or flipping back and forth between the two types of questions. Sometimes, if free response questions are saved for the last part of the session, it is possible students only get practice with one or two of them and most students need additional practice with free response questions.

Analyzing a Function Based on its Derivatives Student Session-Presenter Notes

We have intentionally included more material than can be covered in most Student Study Sessions to account for groups that are able to answer the questions at a faster rate. Use your own judgment, based on the group of students, to determine the order and selection of questions to work in the session. Be sure to include a variety of types of questions (multiple choice, free response, calculator, and non-calculator) in the time allotted. Notice in the solutions guide the questions are categorized as 3, 4, or 5 indicating a typical question of the difficulty level (DL) for a student earning these qualifying scores on the AP exam. This session includes a reference sheet at the back of the packet. We suggest that the presenter does not spend time going over the reference sheet, but may point it out to students who are struggling and also as a general announcement that it is available for students to refer to if needed.

I. 5 Minute Group Activity

- Arrange the students randomly into groups of 3 or 4. Before beginning the questions in the packet, display the graph and questions on the following page for the students- it is also included in their packet. Ask students to discuss the questions while you actively walk around to monitor their progress and listen to their conversations. Use this time to gauge the level of knowledge of the group.
- After a few minutes, discuss the answers to these questions or allow students to come to the board to explain the answers using the provided graph. Try not to spend more than 5 minutes on this activity.

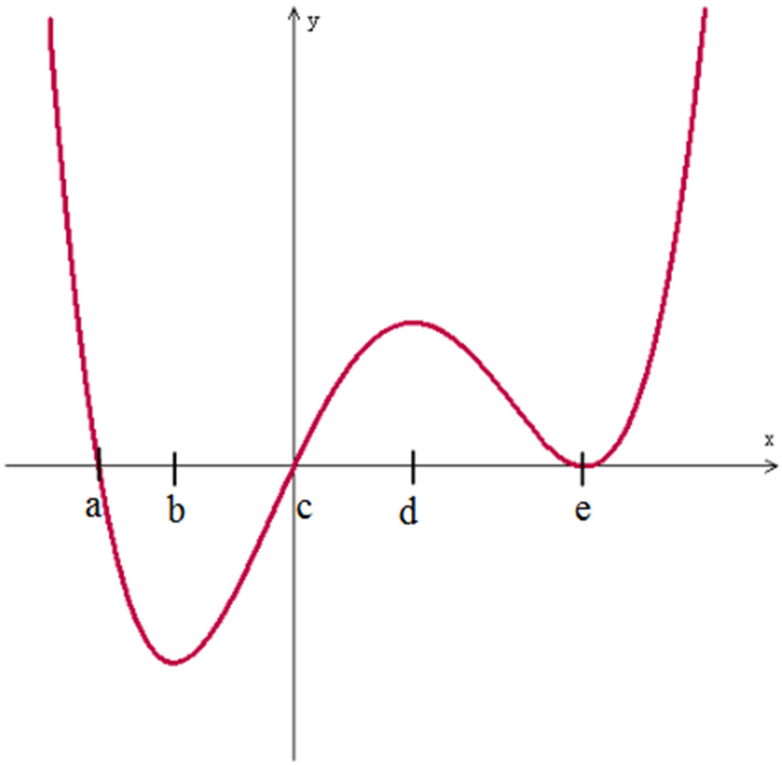
II. 25 minutes Analyzing f, f', and f'' using graph and tables -*your choice of questions depending on student struggles during the first 5 minutes—some suggested questions are listed below.*

- Model several multiple choice questions that involve using graphs or tables, then assign certain questions to the group or pairs. Suggestion: use 2, 9, and 13 as models, then assign 1, 7, 12, 15, and 18.
- Play "pass the pen" to discuss the answers to the assigned problems.
  Pass the Pen: Ask for a student volunteer to begin pass the pen where the multiple choice questions are projected on the board and the student explains how to arrive at the correct answer. The student then selects another student to discuss the next question and so on.
- Allow time for at least one graphical free response question (21, 23)

III. 20 minutes Analyzing f, f', and f'' analytically

- Model several multiple choice questions, then assign certain questions to the group or pairs. Suggestion: use 4, 5, 8, and 10 as models, then assign 3, 6, and 11.
- Allow time for free response question 22.

Connecting $f, f', and\ f''$



Graph of f'

1. On what interval(s) of x is $f(x)$ increasing?

2. On what interval(s) of x is $f(x)$ concave down?

3. At what value(s) of x does $f(x)$ have a relative maximum?

4. At what value(s) of x does $f(x)$ have a point of inflection?

Analyzing a Function Based on its Derivatives Reference Page

Free response questions on this topic often require students to justify their answers. Students may use number lines to analyze the characteristics of the function; however, the justification must be written in sentence form. This justification should avoid using a pronoun such as "it" to describe the function and should make use of calculus involving the first or second derivative test or theorems.

Increase/Decrease

- If $f'(x) > 0$ on an interval, then $f(x)$ is increasing on the interval.
- If $f'(x) < 0$ on an interval, then $f(x)$ is decreasing on the interval.

Relative or Local Extrema – highest or lowest point in the neighborhood

- First derivative test
  - Candidates – critical numbers (x-values that make f' zero or undefined where f is defined)
  - Test – (1) set up an f' number line; label with candidates
    (2) test each section to see if f' is positive or negative
    (3) relative maximum occurs when f' changes from + to –
    relative minimum occurs when f' changes from – to +
- Second derivative test
  - Candidates – critical numbers (x-values that make f' zero or undefined where f is defined)
  - Test – (1) substitute each critical number into the second derivative
    (2) $f'' > 0$, relative minimum
    $f'' < 0$, relative maximum
    (3) $f'' = 0$, the test fails

Absolute or Global Extrema – highest or lowest point in the domain

- Absolute Extrema Test
  - Candidates – critical numbers and endpoints of the domain
  - Test – (1) find the y-values for each candidate
    (2) the absolute maximum value is the largest y-value,
    the absolute minimum value is the smallest y-value

Concavity

- If $f''(x) > 0$ (or $f'(x)$ is increasing) on an interval, then $f(x)$ is concave up on that interval.
- If $f''(x) < 0$ (or $f'(x)$ is decreasing) on an interval, then $f(x)$ is concave down on that interval.

Point of inflection – point where the concavity changes

- Determining points of inflection using the first derivative
  - The graph of f has a point of inflection where f' changes from increasing to decreasing or decreasing to increasing.
- Determining points of inflection using the second derivative
  - Candidates – x-values for which f'' is zero or undefined where f is defined
  - Test – (1) set up an f'' number line; label with candidates
    (2) test each section to see if f'' is positive or negative
    (3) any change in the sign of f'' indicates a point of inflection

<u>Justifications for Increasing/Decreasing Intervals of a function</u>

<u>Remember</u>: $f'(x)$ determines whether a function is increasing or decreasing, so always use the sign of $f'(x)$ when determining and justifying whether a function $f(x)$ is increasing or decreasing on (a, b).

Situation	Explanation
$f(x)$ is increasing on the interval (a, b)	$f(x)$ is increasing on the interval (a, b) because $f'(x) > 0$
$f(x)$ is decreasing on the interval (a, b)	$f(x)$ is decreasing on the interval (a, b) because $f'(x) < 0$

<u>Justifications of Relative Minimums/Maximums and Points of Inflection</u>

Sign charts are very commonly used in calculus classes and are a valuable tool for students to use when testing for relative extrema and points of inflection. However, a sign chart will never earn students any points on the AP exam. Students should use sign charts when appropriate to help make determinations, but they cannot be used as a justification or explanation on the exam.

Situation (at a point $x = a$ on the function $f(x)$)	Proper Explanation/Reasoning
Relative Minimum	$f(x)$ has a relative minimum at the point $x = a$ because $f'(x)$ changes signs from negative to positive when $x = a$.
Relative Maximum	$f(x)$ has a relative maximum at the point $x = a$ because $f'(x)$ changes signs from positive to negative when $x = a$.
Point of Inflection	$f(x)$ has a point of inflection at the point $x = a$ because $f''(x)$ changes sign when $x = a$

Multiple Choice

1. B (1988 AB8) DL: 3

   Since $\frac{dy}{dx} > 0$, $f(x)$ is increasing on the interval and since $\frac{d^2y}{dx^2} < 0$, the graph of $f(x)$ is concave down. Only the interval $b < x < c$ meets both of these criteria.

2. D (1998 AB17) DL: 4

   $f(1) = 0, f'(1) > 0$ since f is increasing, and $f''(1) < 0$ since f is concave down, so answer D provides the correct order for the relationships between f and its derivatives at $x = 1$.

3. C (1998 AB22) DL: 3

   The function, $f(x)$, is increasing when $f'(x) > 0$.

   $f'(x) = 4x^3 + 2x$

   $2x(2x^2 + 1) = 0$

   $x = 0$

   $f'(x) = 4x^3 + 2x$ is positive on the interval $(0, \infty)$.

4. B (1973 AB22) DL: 4

   Determine the interval(s) where $f''(x) > 0$.

   $f'(x) = 15x^4 - 60x^2$

   $f''(x) = 60x^3 - 120x$

   $60x(x^2 - 2) = 0$

   $x = 0$ or $x = \pm\sqrt{2}$

   $f''(x) = 60x^3 - 120x > 0$ on the intervals $\left(-\sqrt{2}, 0\right) \cup \left(\sqrt{2}, \infty\right)$.

5. D (1997 AB22) DL: 5

   Determine the interval(s) where $f'(x) > 0$.

   $f'(x) = (x^2 - 3)(-e^{-x}) + (2x)(e^{-x})$

   $f'(x) = -e^{-x}(x^2 - 2x - 3)$

   $0 = -e^{-x}(x - 3)(x + 1)$

   $x = 3$ and $x = -1$

   $f'(x) > 0$ on $(-1, 3)$

6. C (1998 AB19) DL: 4

   A point of inflection occurs when $f''(x)$ changes sign. In this case, $f''(x)$ has a double root at $x = 2$ and changes sign only at $x = -1$ and $x = 0$.

7. A (2003 AB18) DL: 3

   $g' \le 0$ on this interval

8. C (AP-like) DL: 5

   $\frac{dy}{dx} = f'(x^2 - 4x)(2x - 4)$. Since $y = f(x)$ is decreasing for all real numbers, $f'(x^2 - 4x) < 0$, and $\frac{dy}{dx} = 0$ only at $x = 2$. $\frac{dy}{dx} < 0$ for $x > 2$ and therefore $f(x^2 - 4x)$ is decreasing on the interval $[2, \infty)$.

9. B (2003 AB90) DL: 3

   Answer choices C, D, and E can be eliminated since those choices have a negative first derivative. Only answer B has $f(x)$ values increasing at a decreasing rate.

10. A (AP-like) DL: 5

    f' is positive when $f' > 0$ and f' is decreasing when $f'' < 0$. $f' > 0$ for $x < 0$ and $x > 2$. $f'' < 0$ for $x < 1$. Therefore f' is both positive and decreasing on the interval $(-\infty, 0)$.

11. B (1993 BC22) DL: 3

    The function f is decreasing when f' is negative.

    $f'(x) = x^2 e^x + 2xe^x$

    $xe^x(x + 2) = 0$

    $x = 0$ or $x = -2$

    Thus $f'(x) < 0$ on the interval $(-2, 0)$.

12. C (AB Sample Question #4 from AP Calculus Course and Exam Description) DL:5

13. B (1997 BC80 appropriate for AB) DL: 4

    Using the derivative function on the calculator, graph f'. Calculate the smallest x value where f' changes from increasing to decreasing or decreasing to increasing.

    or

    Using the derivative function on the calculator, graph $f''(x)$. Calculate the smallest x value where $f''(x)$ changes sign.

14. E (1985 BC43 appropriate for AB) DL: 4

    The only graph of f that is continuous, concave down, and matches the first derivative criteria is graph E.

15. C (2003 BC86 appropriate for AB) DL: 4

    Analyze the graph of $f'(x)$ to determine the number of times $f'(x)$ changes from increasing to decreasing or decreasing to increasing on the interval $(-1.8, 1.8)$

    or

    Use the derivative function on the calculator to graph $f''(x)$ to determine the number of times $f''(x)$ changes signs on the interval $(-1.8, 1.8)$

16. C (1997 AB85) DL: 3

    $f'(x)$ changes sign from positive to negative only at $x = 0.91$.

17. B (1998 AB89) DL: 5

    The graph of $y = x^2 - 4$ is a parabola that changes from positive to negative at $x = -2$ and from negative to positive at $x = 2$. Since g is always negative, f' changes sign opposite to the way $y = x^2 - 4$ does. Thus, f has a relative minimum at $x = -2$ and a relative maximum at $x = 2$.

18. E (1998 BC6 appropriate for AB) DL: 3

    $h'(x) > 0$ where $h(x)$ is increasing and $h'(x) < 0$ where $h(x)$ is decreasing. Also, $h'(x) = 0$ where $h(x)$ has a relative maximum and a relative minimum. When $h(x)$ is concave down, $h'(x)$ decreases and when $h(x)$ is concave up, $h'(x)$ increases.

19. E (1997 AB11) DL: 3

    Since $f' < 0$ for $x < -2$ and $x > 2$, the graph of f will be decreasing on these intervals.

    Since $f' > 0$ for $-2 < x < 2$, f is increasing on this interval. Since $f' = 0$ at $x = -2$ and changes from negative to positive, the graph of f has a local minimum at $x = -2$.

    Since $f' = 0$ at $x = 2$ and changes from positive to negative, the graph of f has a local maximum at $x = 2$.

20. D (AP-like) DL: 4

    $f'(x) = 3x^2 + 6x - 9$

    $f'(x) = 3(x+3)(x-1)$

    $f'(x) = 0$ at $x = -3$ and $x = 1$

    f increases for $-4 < x < -3$ and for $1 < x < 2$ and f decreases for $-3 < x < 1$

    The absolute maximum can occur at $x = -3$ or at $x = 2$. $f(-3) = 23$ and $f(2) = -2$

    Therefore, the absolute maximum value is 23.

Free Response

21. 2015 AB5

(a) $f'(x)=0$ at $x=-2$, $x=1$, and $x=4$. $f'(x)$ changes from positive to negative at $x=-2$. Therefore, f has a relative maximum at $x=-2$.	$2\begin{cases}1:\ \text{identifies } x=-2\\ 1:\ \text{answer with reason}\end{cases}$
(b) The graph of f is concave down and decreasing on the intervals $-2<x<-1$ and $1<x<3$ because f' is decreasing and negative on these intervals.	$2\begin{cases}1:\ \text{intervals}\\ 1:\ \text{reason}\end{cases}$
(c) The graph of f has a point of inflection at $x=-1$ and $x=3$ because f' changes from decreasing to increasing at these points. The graph of f has a point of inflection at $x=1$ because f' changes from increasing to decreasing at this point.	$2\begin{cases}1:\ \text{identifies } x=-1, 1, \text{ and } 3\\ 1:\ \text{reason}\end{cases}$

22 2011 AB4 Form B

(a) $f'(x) = 0$ at $x = 4$ $f'(x) > 0$ for $0 < x < 4$ $f'(x) < 0$ for $x > 4$ Therefore f has a relative maximum at $x = 4$.	3 $\begin{cases} 1: \ x = 4 \\ 1: \text{ relative maximum} \\ 1: \text{justification} \end{cases}$
(b) $f''(x) = -x^{-3} + (4-x)(-3x^{-4})$ $= -x^{-3} - 12x^{-4} + 3x^{-3}$ $= 2x^{-4}(x-6)$ $= \dfrac{2(x-6)}{x^4}$ $f''(x) < 0$ for $0 < x < 6$ The graph of f is concave down on the interval $0 < x < 6$.	3 $\begin{cases} 1: \ f''(x) \\ 1: \text{ answer with justification} \end{cases}$

23. 2000 AB3

(a) $x = -1$ $f'(x)$ changes from negative to positive at $x = -1$	2 $\begin{cases} 1: \text{ answer} \\ 1: \text{ justification} \end{cases}$
(b) $x = -5$ $f'(x)$ changes from positive to negative at $x = -5$	2 $\begin{cases} 1: \text{ answer} \\ 1: \text{ justification} \end{cases}$
(c) $f''(x)$ exists and f' is decreasing on the intervals $(-7, -3)$, $(2, 3)$, and $(3, 5)$.	2 $\begin{cases} 1: \ (-7, -3) \\ 1: \ (2, 3) \cup (3, 5) \end{cases}$

