NATIONAL MATH + SCIENCE INITIATIVE

AP Calculus

Theorems (IVT, EVT, and MVT)

Presenter Notes

2017-2018 EDITION

Student Study Session - Presenter Notes

Thank you for agreeing to present at one of NMSI's Saturday Study Sessions. We are grateful you are sharing your time and expertise with our students. Saturday mornings can be a "tough sell" for students, so we encourage you to incorporate strategies and techniques to encourage student movement and engagement. Suggestions for different presentation options are included in this document. If you have any questions about the content or about presentation strategies, please contact Mathematics Director Charla Holzbog at cholzbog@nms.org or AP Calculus Content Specialist Karen Miksch at kmiksch@nms.org.

The material provided contains many released AP multiple choice and free response questions as well as some AP-like questions that we have created. The goal for the session is to let the students experience a variety of both types of questions to gain insight on how the topic will be presented on the AP exam. It is also beneficial for the students to hear a voice other than their teacher in order to help clarify their understanding of the concepts.

Suggestions for presenting:
The vast majority of the study sessions are on Saturday and students and teachers are coming to be WOWed! We want activities to engage the students as well as prepare them for the AP Exam. The following presenter notes include pacing suggestions (you only have 50 minutes!), solutions, and recommended engagement strategies.

Suggestions on how to prepare:

- The notes/summaries on the last page(s) are for reference. We want the students' time during the session to be focused on the questions as much as possible and not taking or reading the notes. As the questions are presented during the session, you may wish to refer the students back to those pages as needed. It is not our intent for the sessions to begin with a lecture over these pages.
- As you prepare, work through the questions in the packet noting the level of difficulty and topic or skill required for the questions.
- Design a plan for what questions you would like to cover with the group depending on their level of expertise. Some groups will be ready for the tougher questions while other groups will need more guidance and practice on the easier ones. Create an easy, medium, and hard listing of the questions prior to the session. This will allow you to adjust on the fly as you get to know the groups. In most instances, there will **not** be enough time to cover all the questions in the packet. Use your judgement on the amount of questions to cover based on the students' interactions. Remember to include both multiple choice and free response type questions. Discussions on test taking strategies and scoring of the free response questions are always great to include during the day.
- The concepts should have been previously taught; however, be prepared to "teach" the topic if you find out the students have not covered the concept prior in class. In sessions where multiple schools come together, you might have a mixture of students with and without prior knowledge on the topic. You will have to use your best judgement in this situation.
- Consider working through some free response questions before the multiple choice questions, or flipping back and forth between the two types of questions. Sometimes, if free response questions are saved for the last part of the session, it is possible students only get practice with one or two of them and most students need additional practice with free response questions.

Theorems Student Session-Presenter Notes

We have intentionally included more material than can be covered in most Student Study Sessions to account for groups that are able to answer the questions at a faster rate. Use your own judgment, based on the group of students, to determine the order and selection of questions to work in the session. Be sure to include a variety of types of questions (multiple choice, free response, calculator, and non-calculator) in the time allotted. Notice in the solutions guide the questions are categorized as 3, 4, or 5 indicating a typical question of the difficulty level (DL) for a student earning these qualifying scores on the AP exam. This session includes a reference sheet at the back of the packet. We suggest that the presenter does not spend time going over the reference sheet, but may point it out to students who are struggling and also as a general announcement that it is available for students to refer to if needed.

I. 10 Minute Group Activity

- Arrange the students randomly into groups of 3 or 4. Ask students to work together on free response question 16 while you actively walk around to monitor their progress and assist with any questions. Use this time to gauge the level of knowledge of the group.
- After a few minutes, review the answers with students and clarify any misunderstandings. You may even want to show the students the rubric and discuss what is necessary to earn points.

II. 20 minutes Multiple Choice Practice-*your choice of questions depending on student struggles during the first ten minutes—some suggested questions are listed below.*

- Choose a few multiple choice questions to model for the students, and then choose several to assign to the groups. Selections should be based on the knowledge level of the group. Use a think-pair-share strategy for the assigned questions.
- Suggestion: use 2, 3, and 10 as models, then assign 1, 6, 7, 8, and 11 (include 5 and 10 for stronger groups).

III. 20 minutes additional Free Response Practice

- Assign questions 12 and 13 to the groups. Allow time for some discussion within the groups before reviewing the answers with the students.
- Assign remaining free response questions (14, 15, and 17) as time permits.

Multiple Choice Solutions

1. B (1998 AB4) DL: 3
   B is false because this special case of the MVT called Rolle's Theorem also requires that $f(a) = f(b)$. A is true by MVT; C and D are true by EVT, E is true by the definition of a definite integral.

2. D (AP-like) DL: 4
   Show $f'(c) = \frac{f(b) - f(a)}{b - a}$ using MVT: $\frac{2x^2 - 5}{x^2} = \frac{11 - 7}{5 - 1} = 1$; $2x^2 - 5 = x^2$; $x^2 - 5 = 0$ when $x = -\sqrt{5}$ and $x = \sqrt{5}$; however, only $\sqrt{5}$ is eligible since $x = -\sqrt{5}$ is not in the given interval.

3. A (1998 AB26) DL: 3
   Any value of k less than $\frac{1}{2}$ will require the function to assume the value of $\frac{1}{2}$ at least twice because of the Intermediate Value Theorem on the intervals $[0, 1]$ and $[1, 2]$, so $k = 0$ is the only option.

4. D (1973 BC18 appropriate for AB) DL: 3
   D could be false, consider $g(x) = 1 - x$ on $[0, 1]$.
   A is true by the Extreme Value Theorem.
   B is true because g is a function.
   C is true by the Intermediate Value Theorem.
   E is true because g is continuous.

5. D (1993 AB18) DL: 5

$$f'(c)=\frac{1}{2}\cos\left(\frac{c}{2}\right);\ f'(c)=\frac{f\left(\frac{3\pi}{2}\right)-f\left(\frac{\pi}{2}\right)}{\frac{3\pi}{2}-\frac{\pi}{2}}=\frac{\sin\left(\frac{3\pi}{4}\right)-\sin\left(\frac{\pi}{4}\right)}{\frac{3\pi}{2}-\frac{\pi}{2}}=\frac{0}{\pi}=0;$$

$$\frac{1}{2}\cos\left(\frac{c}{2}\right)=0;\ c=\pi$$

6. D (AP-like) DL: 3
   The Mean Value Theorem requires differentiability so only IVT and EVT apply.

7. B (AP-like) DL: 3

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$f'(c)=\frac{f(2)-f(0)}{2-0}=\frac{7-1}{2}=3$$

8. C (AP-like) DL: 3
   f must be differentiable on the open interval for MVT to apply

9. C (AP-like) DL: 4
   II is true by IVT and III is true by MVT

10. B (2003 AB80) DL: 4
    B could be false since this is a special case of MVT (Rolle's Theorem) which also requires that $f(a)=f(b)$. A and C are true by IVT; D is true by MVT; E is true by EVT.

11. C (AP-like) DL: 3
    $g(x)=2$ at least once in the interval $(0,2)$ and at least once in the interval $(2,5)$ and at least once in the interval $(9,11)$. Therefore, $g(x)=2$ at least three times.

12. 2013 AB3b

(b) C is differentiable $\Rightarrow$ C is continuous (on the closed interval) $\frac{C(4)-C(2)}{4-2}=\frac{12.8-8.8}{2}$ Therefore, by the Mean Value Theorem, there is at least one time t, $2<t<4$, for which $C'(t)=2$.	$2\begin{cases}1: \frac{C(4)-C(2)}{4-2}\\ 1: \text{conclusion, using MVT}\end{cases}$

13. 2014 AB4b

(b) v_A is differentiable $\Rightarrow$ v_A is continuous $v_A(8)=-120<-100<40=v_A(5)$ Therefore, by the Intermediate Value Theorem, there is a time t, $5<t<8$, such that $v_A(t)=-100$.	$2\begin{cases}1: v_A(8)<-100<v_A(5)\\ 1: \text{conclusion, using IVT}\end{cases}$

14. 2009B AB3c

(c) Yes, $a=3$. The function f is differentiable on the interval $3<x<6$ and continous on $3\le x\le 6$. Also, $\frac{f(6)-f(3)}{6-3}=\frac{1-0}{6-3}=\frac{1}{3}$. By the Mean Value Theorem, there is a value *c*, $3<x<6$, such that $f'(c)=\frac{1}{3}$.	$2\begin{cases}1: \text{answers “yes” and identifies } a=3\\ 1: \text{justification}\end{cases}$

15. 2008B AB5/BC5d

(d) $\frac{g'(7)-g'(-3)}{7-(-3)}=\frac{1-(-4)}{10}=\frac{1}{2}$ No, the MVT does *not* guarantee the existence of a value *c* with the stated properties because g' is not differentiable for at least one point in $-3<x<7$.	$2\begin{cases}1: \text{average value of } g'(x)\\ 1: \text{answer “No” with reason}\end{cases}$

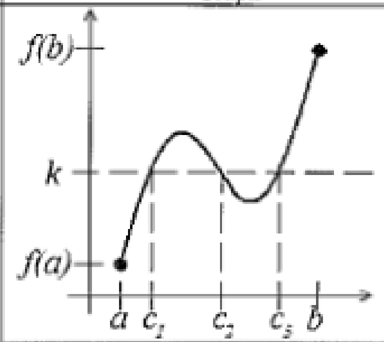
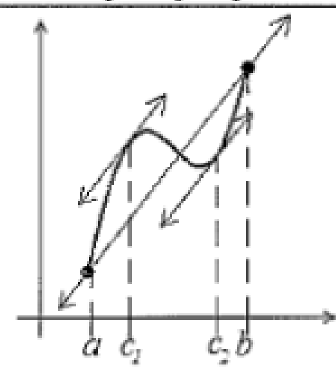
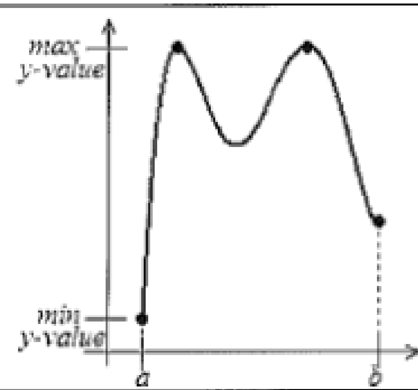
16. 2007 AB 3ab

(a) $h(1) = f(g(1)) - 6 = f(2) - 6 = 9 - 6 = 3$ $h(3) = f(g(3)) - 6 = f(4) - 6 = -1 - 6 = -7$ Since $h(3) < -5 < h(1)$ and h is continuous, by the Intermediate Value Theorem, there exists a value r, $1 < r < 3$, such that $h(r) = -5$.	2: 1: $h(1)$ and $h(3)$ 1: conclusion, using IVT
(b) $\frac{h(3)-h(1)}{3-1} = \frac{-7-3}{3-1} = -5$ Since h is continuous and differentiable, by the Mean Value Theorem, there exists a value c, $1 < c < 3$, such that $h'(c) = -5$.	2: 1: $\frac{h(3)-h(1)}{3-1}$ 1: conclusion, using MVT

17. 2007B AB 6abd

(a) The Mean Value Theorem guarantees that there is a value c, with $2 < c < 5$, so that $f'(c) = \frac{f(5)-f(2)}{5-2} = \frac{2-5}{5-2} = -1.$	2: 1: $\frac{f(5)-f(2)}{5-2}$ 1: conclusion, using MVT
(b) $g'(x) = f'(f(x)) \cdot f'(x)$ $g'(2) = f'(f(2)) \cdot f'(2) = f'(5) \cdot f'(2)$ $g'(5) = f'(f(5)) \cdot f'(5) = f'(2) \cdot f'(5)$ Thus, $g'(2) = g'(5)$. Since *f* is twice-differentiable, g' is differentiable everywhere, so the Mean Value Theorem applied to g' on [2, 5] guarantees there is a value k, with $2 < k < 5$, such that $g''(k) = \frac{g'(5)-g'(2)}{5-2} = 0.$	3: 1: $g'(x)$ 1: $g'(2) = f'(5) \cdot f'(2) = g'(5)$ 1: uses MVT with g'
(d) Let $h(x) = f(x) - x$. $h(2) = f(2) - 2 = 5 - 2 = 3$ $h(5) = f(5) - 5 = 2 - 5 = -3$ Since $h(2) > 0 > h(5)$, the Intermediate Value Theorem guarantees that there is a value r, with $2 < r < 5$, such that $h(r) = 0$	2: 1: $h(2)$ and $h(5)$ 1: conclusion, using IVT

Reference Page

Name	Formal Statement	Restatement	Graph	Notes
IVT	If $f(x)$ is continuous on a closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$, then there exists at least one value c in $[a, b]$ such that $f(c)=k$.	On a continuous function, you will hit every y-value between two given y-values at least once.		When writing a justification using the IVT, you must state the function is continuous even if this information is provided in the question.
MVT	If $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on (a, b), then there must exist at least one value c in (a, b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$	If conditions are met (very important!) there is at least one point where the slope of the tangent line equals the slope of the secant line.		When writing a justification using the MVT, you must state the function is differentiable (continuity is implied by differentiability) even if this information is provided in the question. (Questions may ask students to justify why the MVT cannot be applied often using piecewise functions that are not differentiable over an open interval.)
EVT	A continuous function $f(x)$ on a closed interval $[a, b]$ attains both an absolute maximum $f(c) \geq f(x)$ for all x in the interval and an absolute minimum $f(c) \leq f(x)$ for all x in the interval	Every continuous function on a closed interval has a highest y-value and a lowest y-value.		When writing a justification using the EVT, you must state the function is continuous on a closed interval even if this information is provided in the question.