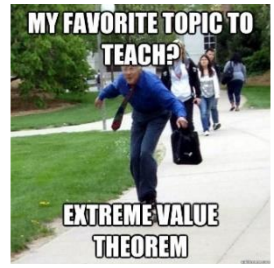
AB Calculus: Extreme Values of a Function **Name:** ______________________________

Extrema (plural for extremum) are the maximum and minimum values of a function. In the past, you have used your calculator to calculate the maximum and minimum value. In this section, you will learn to use calculus reasons to find extrema, how to distinguish between absolute extrema and relative extrema, and how to locate them.



Definition of Absolute Extrema … the BIGGEST or smallest y-value in the interval

Let f be defined on an interval I containing c

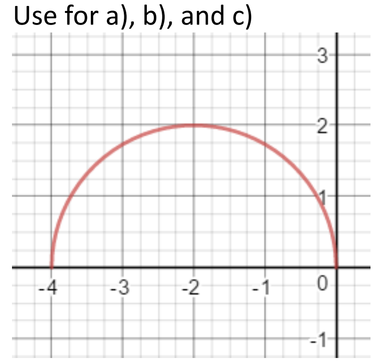
1. $f(c)$ is the minimum of f on the interval I if $f(c) \leq f(x)$ for all x in I.
2. $f(c)$ is the maximum of f on the interval I if $f(c) \geq f(x)$ for all x in I.

The minimum and maximum of a function on an interval are the **extreme values**, or **extrema**, of the function on the interval. The minimum and maximum of a function on an interval are also called the **absolute minimum** and **absolute maximum**, or the **global minimum** and **global maximum**, on the interval. Extrema can occur at interior points or endpoints of an interval.

A function may have both a maximum and a minimum value over an interval, only a maximum, only a minimum, or neither a maximum or minimum value of an interval. Try the following example to see some reasons why.

Example 1 Find the absolute extrema of the functions over each given interval. Use the top graph for parts a,b, and c and use the bottom graph for part d.

Use for a), b), and c)



a) $[-4, 0]$ *closed interval

abs max = 2 at x = -2

abs min = 0 at x = -4 and x = 0

b) $[-2, 0)$

abs max = 2 at x = -2

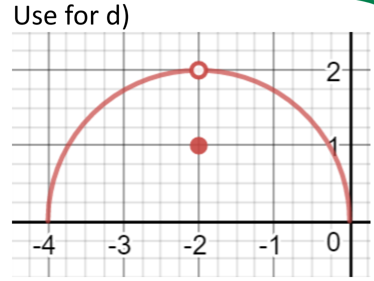
no abs min

c) $(-4, -2)$

no abs max

no abs min

Use for d)



d) $[-4, 0]$

no abs max

abs min = 0 at x = -4 and x = 0

What two things needed to be the case for there to be both a minimum and a maximum for the graphs in Ex. 1?

closed and continuous

The Extreme Value Theorem

If f is continuous over a closed interval $[a, b]$, then f has both a minimum and a maximum over the interval.

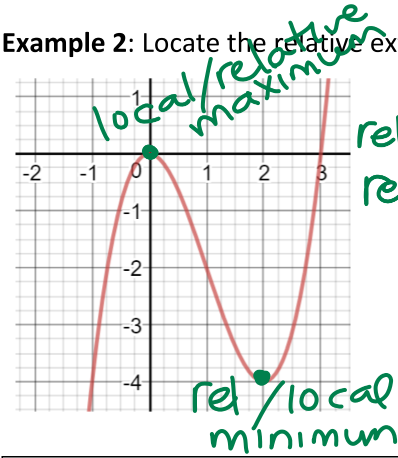
Relative Extrema and Critical points

Definition of Relative Extrema

1. If there is an open interval containing a c on which $f(c)$ is a maximum, then $f(c)$ is called a **relative maximum** or **local maximum**. You could also say that f has a relative maximum at $(c, f(c))$.
2. If there is an open interval containing a c on which $f(c)$ is a minimum, then $f(c)$ is called a **relative minimum** or **local minimum**. You could also say that f has a relative minimum at $(c, f(c))$.

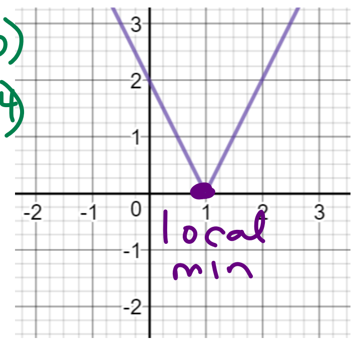
Basically, relative extrema exist when the value of the function is larger (or smaller) than all other function values relatively close to that value.

Example 2: Locate the relative extrema in each graph below and determine the value of the derivative.



Relative Extrema: rel max is at (0,0) rel min is at (2,-4)

What is the value of the derivative at each Relative Extrema? $\frac{dy}{dx} = 0$



Relative Extrema: local min at (1,0)

What is the value of the derivative at each Relative Extrema? $\frac{dy}{dx}$ is undefined

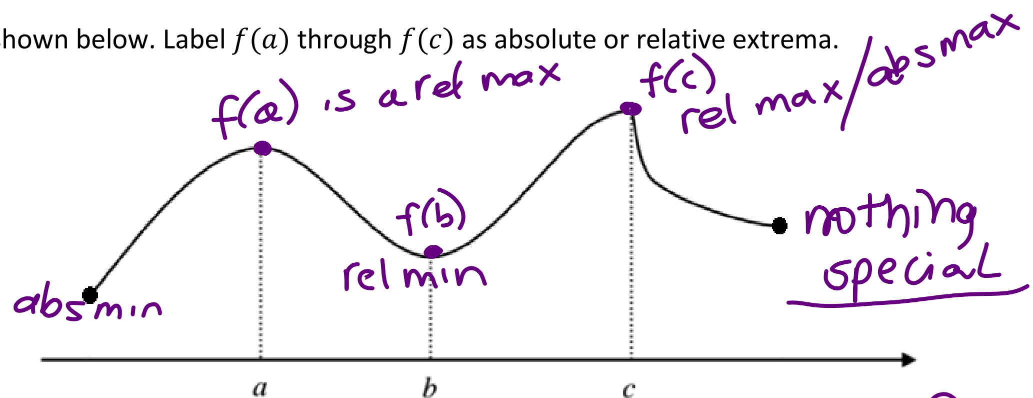
Definition of Critical Point

Let f be defined at c. If $f'(c) = 0$ or if $f'(c)$ is undefined, then c is a **critical point** of f.

Relative Extrema only occur at critical points

If f has a relative max or relative min at $x = c$, then c is a critical point of f.

Example 3 the graph of $f(x)$ is shown below. Label $f(a)$ through $f(c)$ as absolute or relative extrema.



When given a graph it is fairly simple to identify the extrema. The question to be asked then is how do we find the extrema when we do not have a graph given to us.

Important Note: Just because the derivative is equal to zero (or undefined) does not mean there is a relative maximum or minimum at the point. The sign of the derivative needs to change sign for a maximum or minimum to exist at that point. This does not happen at every critical point, but it can only happen at critical points.

Guidelines for Finding Absolute/Relative Extrema on a Closed Interval

1. Find the critical numbers of f on (a, b). Do this by setting the derivative equal to 0 and undefined. These critical points and the endpoints make up the list of candidates for the extrema.
2. Evaluate each candidate by plugging these numbers into the original function.
3. The least of the values from the previous step is the absolute minimum, and the greatest of these values is the absolute maximum.

The critical points are $\underline{\ x\ }$ values, while maximums/minimums of the function are $\underline{\ y\ }$ values. In other words, if the point $(2, 70)$ is a relative minimum, the minimum of the function is $\underline{\ 70\ }$ and it occurs at $\underline{\ 2\ }$.

Example 4 Find the absolute extrema of $f(x) = 3x^4 - 4x^3$ over the interval $[-1, 2]$.

$f'(x) = 12x^3 - 12x^2$

$0 = 12x^3 - 12x^2$

$0 = 12x^2(x - 1)$

$x = 0$ $\quad$ $x = 1$

x	$f(x)$
-1	7
0	0
1	-1
2	16

$$\boxed{\begin{aligned}&\text{abs max} = 16 \text{ at } x = 2\\ &\text{abs min} = -1 \text{ at } x = 1\end{aligned}}$$

Example 5 Find the extrema of $f(x) = 2x - 3x^{2/3}$ over the interval $[-1, 3]$.

$f'(x) = 2 - 2x^{-\frac{1}{3}}$

$0 = 2 - \dfrac{2}{\sqrt[3]{x}}$

$\dfrac{2}{\sqrt[3]{x}} = 2$

$(2 = 2\sqrt[3]{x})^3$

$8 = 8x$

$1 = x$

$f'(x)$ is undefined at $x = 0$ → critical pts.

$f'(x) = 0$ at $x = 1$

x	$f(x)$
-1	-5
0	0
1	-1
3	$6 - 3\sqrt[3]{9} < 0$

$$\boxed{\begin{aligned}&\text{abs max} = 0 \text{ at } x = 0\\ &\text{abs min} = -5 \text{ at } x = -1\end{aligned}}$$

Example 6 Find the absolute extrema of $f(x) = \frac{2x}{x^2+1}$ over the interval $[-2, 2]$.

$$f'(x) = \frac{(x^2+1)(2) - (2x)(2x)}{(x^2+1)^2}$$

$$= \frac{2x^2 + 2 - 4x^2}{(x^2+1)^2}$$

$$= \frac{-2x^2 + 2}{(x^2+1)^2}$$

$$\frac{-2(x^2-1)}{(x^2+1)^2}$$

$$\frac{-2(x+1)(x-1)}{(x^2+1)^2}$$

$f'(x) = 0$ at $x = -1, 1$ → critical pts

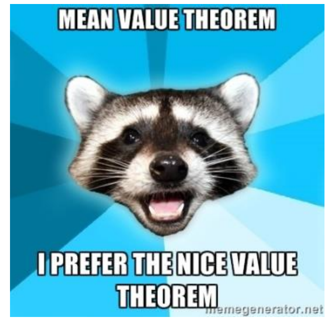
x	$f(x)$
-2	$-4/5$
-1	-1
1	1
2	$4/5$

abs max $= 1$ at $x = 1$

abs min $= -1$ at $x = -1$

AB Calculus: The Mean Value Theorem **Name: ______________________________**

The Mean Value Theorem (MVT) is considered by some to be the most important topic in all of calculus other than limits. It is used to prove many of the theorems in calculus that we use in this course as well as further studies into calculus.

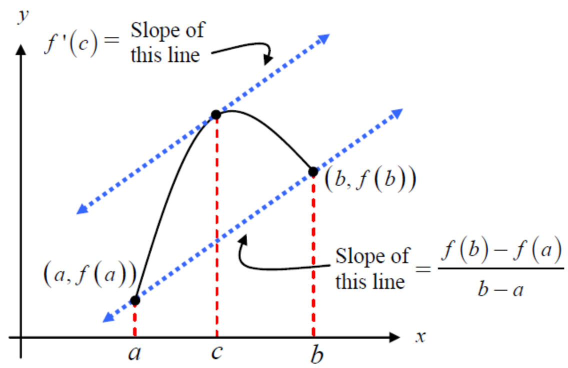


The Mean Value Theorem

If f is continuous over the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

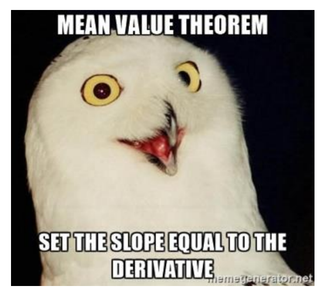
Some important notes regarding the Mean Value Theorem



- Just like the Intermediate Value Theorem, this is an existence theorem. The Mean Value Theorem does not tell you what the value of c is, nor does it tell you how many exist.
- The hypothesis of the Mean Value Theorem is highly important. If any part of the hypothesis does not hold, the theorem cannot be applied.
- Remember, c is an x-value.
- Basically, the Mean Value Theorem says that the average rate of change over the entire interval is equal to the instantaneous rate of change at some point inside the interval.

Example 1 Apply the Mean Value Theorem to the function on the indicated interval. In each case, make sure the hypothesis is true, then find all values of c inside the interval guaranteed by the MVT.

a) $f(x) = x(x^2 - x - 2)$ over the interval $[-1, 1]$



b) $f(x) = \frac{x+5}{x-1}$ over the interval $[-3, 5]$

Increasing vs. Decreasing

Definitions of Increasing and Decreasing Functions

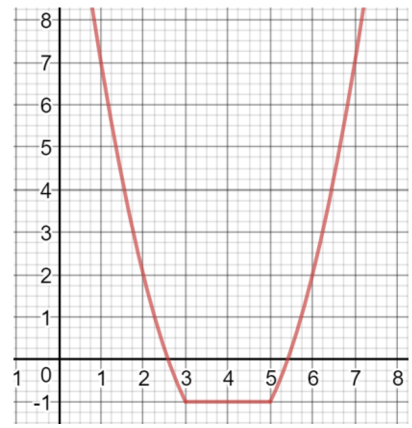
A function f is increasing on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

A function f is decreasing on an interval if for any two numbers x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

Example 2

a) Over what interval is the function increasing? decreasing? constant?

b) What is the value of the derivative when the function is increasing? decreasing? constant?



Test for Increasing and Decreasing Functions

Let f be a function that is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b).

1. If $f'(x) > 0$ for all x in (a, b), then f is increasing on (a, b)
2. If $f'(x) < 0$ for all x in (a, b), then f is decreasing on (a, b)
3. If $f'(x) = 0$ for all x in (a, b), then f is constant on (a, b)

Guidelines for Finding Intervals on which a function is Increasing or Decreasing

Let f be a function that is continuous on the closed interval $[a, b]$. To find the intervals on which f is increasing or decreasing use the following steps:

1. Find the critical points of f in the interval (a, b) and use these numbers to create a sign chart.
2. Use the signs of the derivative to determine whether the function is increasing or decreasing.

Your sign chart is NOT a justification to your response. Your response should be written in words …
"The function is increasing (or decreasing) on the interval (c, d) since $f'(x) > 0$ (or $f'(x) < 0$)."

Example 3 Find the intervals on which $f(x) = 4x^3 - 15x^2 - 18x + 7$ is increasing or decreasing. Also, find all local extrema and justify each response.

Example 4 Find the intervals on which $f(x) = (x^2 - 9)^{\frac{2}{3}}$ is increasing or decreasing. Also, find all local extrema and justify each response.

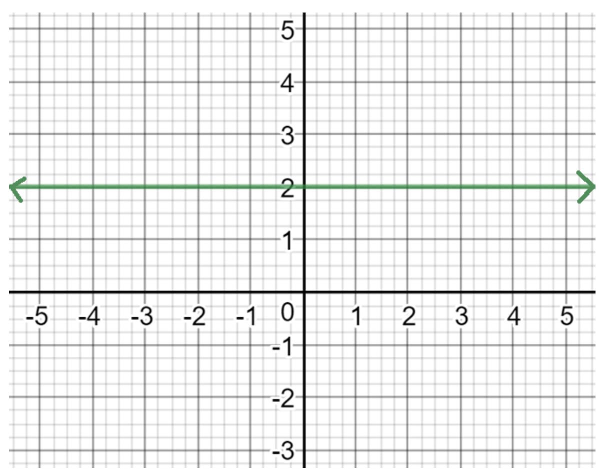
Anti-Derivatives

Example 5 Suppose you were told that $f'(x) = 2x - 1$.

a) What could $f(x)$ be? Is there more than one answer?

The process of finding the function from the derivative is called **antidifferentiation**.

b) Suppose the graph of $f'(x)$ is given to the right. Draw at least 3 possibilities for $f(x)$. Remember, if $f'(x)$ is given, then the y-values are the slopes of the original function at those x-values.



The three functions you drew should only differ by a constant (differ by a vertical transformation). If you let C represent this constant, then you can represent the family of all antiderivatives of $f'(x)$ to be $f(x) = 2x + C$.

c) If you were told that $f(3) = -2$, what would the value of C be?

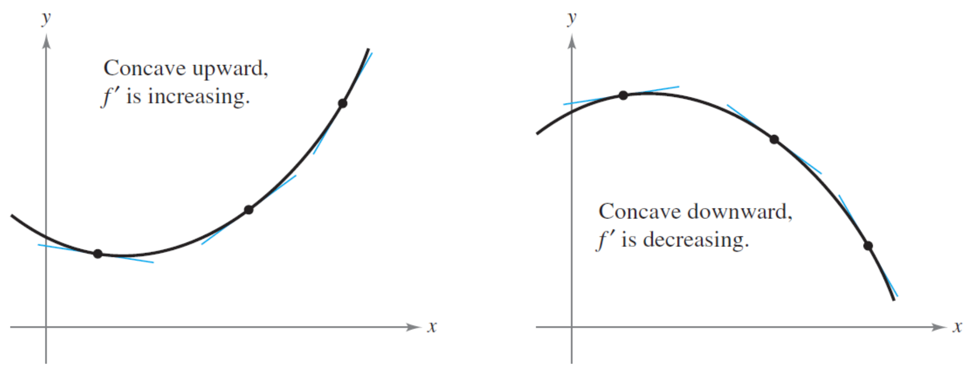
If a function has one antiderivative then it has many antiderivatives that all differ by a constant. Unless you know something about the original function, you cannot determine the exact value of that constant, but the $+C$ must be included in your answer.

AB Calculus: Concavity and the Second Derivative Test   Name: ______________________________

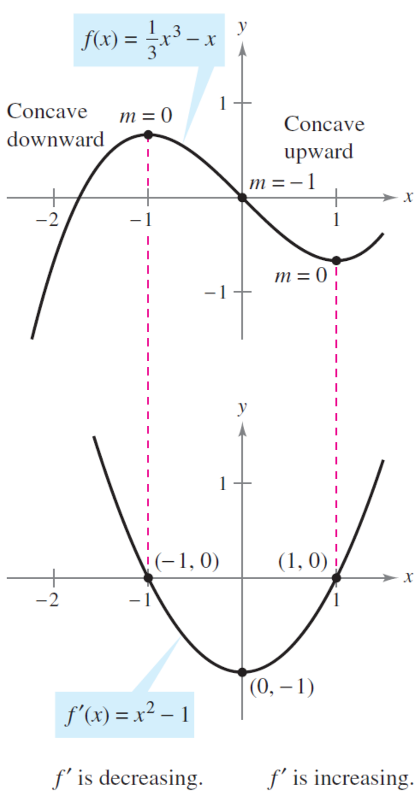
You have already seen that locating the intervals in which a function is increasing and decreasing helps to describe its graph. In this section, you will see how locating the intervals in which the derivative is increasing or decreasing can be used to determine when a function is curving upward or curving downward.

Definition: Concavity

Let f be differentiable on an open interval I. The graph of f is concave upward on I if f' is increasing on the interval and concave downward on I if f' is decreasing on the interval.



(a) The graph of f lies above its tangent lines. **(b)** The graph of f lies below its tangent lines.



To find the open intervals on which the graph of a function f is concave upward or concave downward, you need to find the intervals on which f' is increasing or decreasing. For instance, the graph of

$$f(x) = \frac{1}{3}x^3 - x$$

Is concave downward over the open interval $(-\infty, 0)$ because $f'(x) = x^2 - 1$ is decreasing there. Similarly, the graph of f is concave upward on the interval $(0, \infty)$ because f' is increasing over $(0, \infty)$.

The following theorem shows how to use the second derivative of a function f to determine intervals on which the graph of f is concave upward or concave downward.

Test for Concavity

Let f be a function whose second derivative exists on an open interval I.

1. If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.
2. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

- If the second derivative is positive, then the first derivative is _________________________ and the original function is ______________________________.
- If the second derivative is negative, then the first derivative is _________________________ and the original function is ______________________________.

Guidelines for Determining Concavity

1. Find the second derivative of the function.
2. Locate the x-values where $f''(x) = 0$ and $f''(x)$ is undefined.
3. Make a sign chart for the second derivative using the x-values you found in part 2.
4. In areas where the second derivative is positive, $f(x)$ is concave up. In areas where $f''(x)$ is negative, $f(x)$ is concave down.

Example 1 Let f be the function defined below. Determine the open intervals on which the graph of f is concave upward, then find the intervals on which f is concave downward.

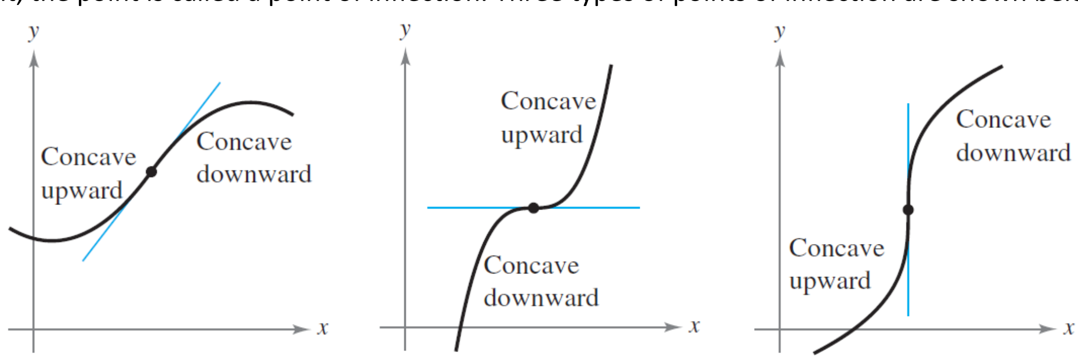
$$f(x) = 4x^3 + 21x^2 + 36x - 20$$

Example 2 Let f be the function defined below. Determine the open intervals on which the graph of f is concave upward, then find the intervals on which f is concave downward.

$$f(x) = \frac{x^2 + 1}{x^2 - 4}$$

Points of Inflection

The graph in example 2 has two points at which the concavity changes. If the tangent line to the graph exists at such a point, the point is called a point of inflection. Three types of points of inflection are shown below.



Definition: Point of Inflection (POI)

Let f be a function that is continuous on an open interval and let c be a point in the interval. If the graph of f has a tangent line at this point $(c, f(c))$, then this point is a point of inflection of the graph of f if the concavity of f changes from upward to downward or downward to upward at the point.

If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or $f''(c)$ is undefined.

Example 3: Determine the points of inflection of the graph of $f(x) = x^4 - 4x^3$

The Second Derivative Test

In addition to testing for concavity, the second derivative can be used to perform a simple test for relative maxima and minima. The test is based on the fact that if the graph of a function f is concave upward on an open interval containing c, and $f'(c) = 0$, $f(c)$ must be a relative minimum. Similarly, if the graph of a function f is concave downward on an open interval containing c, and $f'(c) = 0$, $f(c)$ must be a relative maximum of f.

The Second Derivative Test

Let f be a function such that $f'(c) = 0$ and the second derivative of f exists on an open interval containing c.

1. If $f''(c) > 0$, then f has a relative minimum at $(c, f(c))$.
2. If $f''(c) < 0$, then f has a relative maximum at $(c, f(c))$

If $f''(c) = 0$, the test fails. That is, f may have a relative maximum, a relative minimum, or neither. In such cases, you can use the first derivative test.

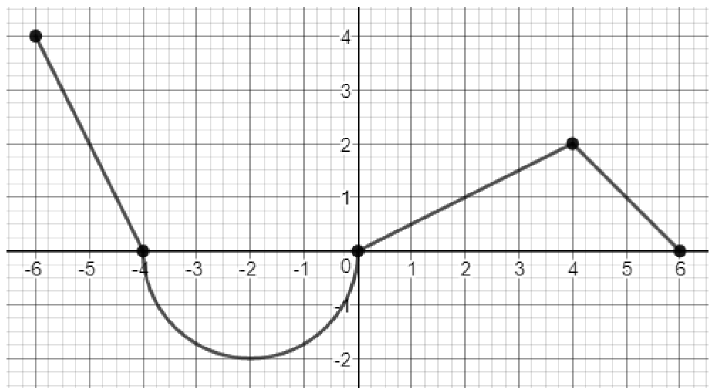
Example 4: Find the relative extrema for the function $f(x) = -3x^5 + 5x^3$ using the second derivative test.

Note: Usually, you can choose whether you use the first derivative test to find relative extrema or the second derivative test. However, there are ways to ask a question so that you have to use the second derivative test. For example, being given that $f(x)$ is continuous at $x = 2$, $f'(2) = 0$, and $f''(2) = -3$ is enough information to determine that $f(x)$ has a relative max at $x = 2$ because of the second derivative test. You would be unable to make that determination if you only knew the first derivative test.

AB Calculus: Information About f(x) Given f'(x) **Name: ____________________**

In this section, we are going to look at the relationships we have previously learned about a function, its derivative, and its second derivative through another perspective. We have already been given either the graph of the original function or the equation for the original function and been able to find information about the original function such as: intervals the function is increasing or decreasing, x-coordinates of relative extrema, intervals the function is concave up or concave down, and x-coordinates of points of inflection. Today, we will learn how to find this same information given the graph of the derivative.

Example 1: The function f is defined and differentiable on the closed interval $[-6, 6]$. The graph of $y = f'(x)$, the derivative of f, consists of a semicircle and three line segments, as shown in the figure below.



a) Find the x-coordinate of each critical point of $y = f(x)$ on the interval $-6 < x < 6$. Justify your answer.

b) Find the x-coordinate of each relative extrema of $y = f(x)$ on the interval $-6 < x < 6$. Label each as a minimum or maximum and justify each response.

c) Find the open intervals over which the function $y = f(x)$ is increasing and decreasing on the interval $-6 < x < 6$. Justify your answer.

d) Find the x-coordinate of each point of inflection for $y = f(x)$ on the interval $-6 < x < 6$. Justify your answer.

e) Find the intervals over which the function $y = f(x)$ is concave up and concave down on the interval $-6 < x < 6$. Justify your answer.

f) If $f(-2) = 4$, find the equation of the tangent line to $y = f(x)$ at $x = -2$.

Guidelines for Finding Information From the Graph of the Derivative of a Function

Critical points for the function $y = f(x)$ occur when the graph of $y = f'(x)$:

Relative maxima for the function $y = f(x)$ occur when the graph of $y = f'(x)$:

Relative minima for the function $y = f(x)$ occur when the graph of $y = f'(x)$:

The function $y = f(x)$ is increasing when the graph of $y = f'(x)$:

The function $y = f(x)$ is decreasing when the graph of $y = f'(x)$:

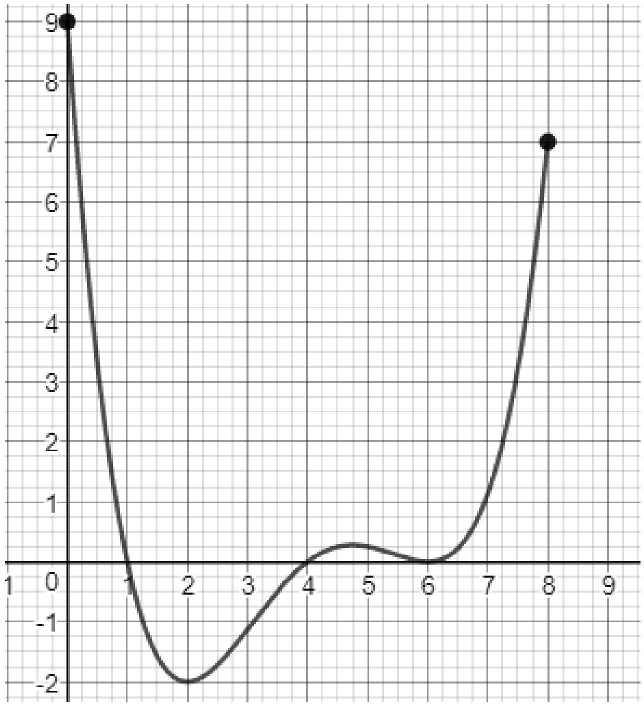
Points of inflection for the function $y = f(x)$ occur when the graph of $y = f'(x)$:

The function $y = f(x)$ is concave up when the graph of $y = f'(x)$:

The function $y = f(x)$ is concave down when the graph of $y = f'(x)$:

The slope of the original function at a point is:

Example 2: The function f is defined and differentiable on the closed interval $[0, 8]$. The graph of $y = f'(x)$, the derivative of f, is shown in the figure below and has horizontal tangents at $x = 2$, $x = 4.75$, and $x = 6$.



a) Find the x-coordinate of each critical point of $y = f(x)$ on the interval $0 < x < 8$. Justify your answer.

b) Find the open intervals over which the function $y = f(x)$ is increasing and decreasing on the interval $0 < x < 8$. Justify your answer.

c) Find the x-coordinate of each relative extrema of $y = f(x)$ on the interval $0 < x < 8$. Label each as a minimum or maximum and justify each response.

d) Find the x-coordinate of each point of inflection for $y = f(x)$ on the interval $0 < x < 8$. Justify your answer.

e) Find the intervals over which the function $y = f(x)$ is concave up and concave down on the interval $0 < x < 8$. Justify your answer.

f) Let $g(x) = x^2 - f(x)$. Find $g'(2)$.

AB Calculus: Curve Sketching **Name:** ______________________________

In this section, we are going to combine everything we have previously learned to sketch functions without a calculator.

Guidelines for Analyzing the Graph of a Function
1. Determine the intercepts, asymptotes, and symmetry of the graph. 2. Locate the x-values for which $f'(x)$ and $f''(x)$ are either zero or undefined. 3. Use the results to determine relative extrema and points of inflection as well as intervals that the function is increasing and decreasing 4. Sketch the graph so that it supports all the information you found.

Example 1 Analyze and sketch the graph of $f(x) = (x-2)^2(x+1)$

a) Determine the x- and y-intercepts

b) Determine the equations of any horizontal and vertical asymptotes.

c) Determine the first and second derivative of the function.

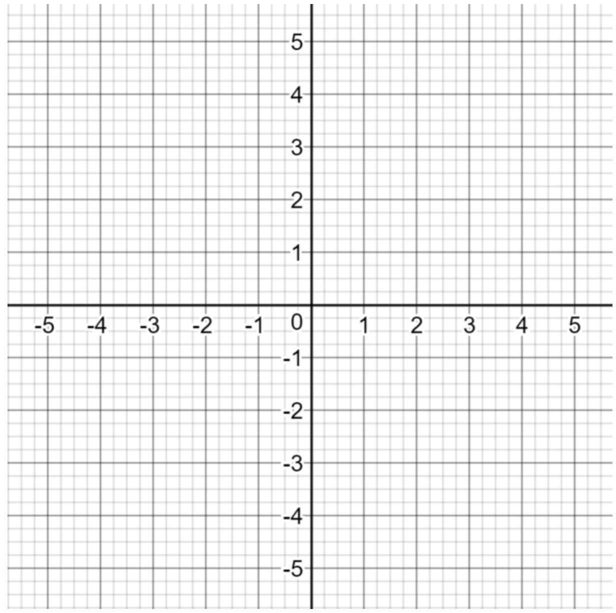
d) Complete a sign chart for $f'(x)$ to determine the intervals where the function is increasing/decreasing.

e) Determine any relative minimums or maximums.

f) Complete a sign chart for $f''(x)$ to determine the intervals where the function is concave up/down.

g) Identify any points of inflection.

h) Complete a sketch of the function that supports all of the information above.



Example 2 Analyze and sketch the graph of $f(x) = \frac{(x^2-9)}{x^2-4}$

a) Determine the x- and y-intercepts

b) Determine the equations of any horizontal and vertical asymptotes.

c) Determine the first and second derivative of the function.

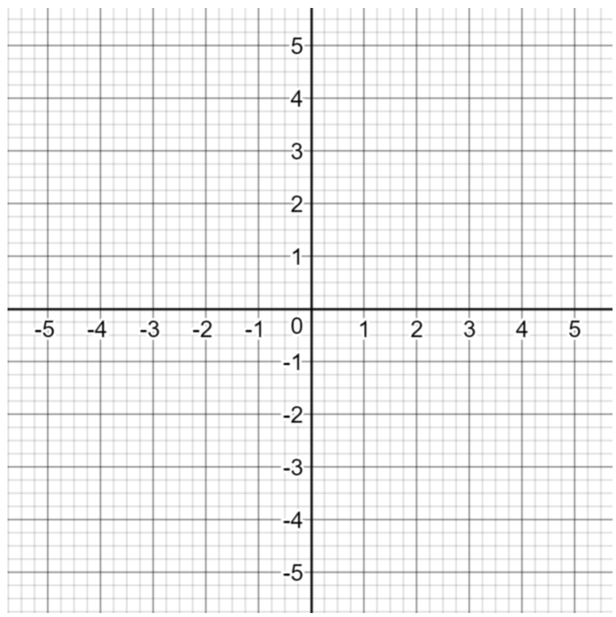
d) Complete a sign chart for $f'(x)$ to determine the intervals where the function is increasing/decreasing.

e) Determine any relative minimums or maximums.

f) Complete a sign chart for $f''(x)$ to determine the intervals where the function is concave up/down.

g) Identify any points of inflection.

h) Complete a sketch of the function that supports all of the information above.



How to Draw Sections of a Curve			
Increasing Concave Up	**Increasing Concave Down**	**Decreasing Concave Up**	**Decreasing Concave Down**

Example 3 Sketch a curve illustrating a function such that

$f(-2) = 8$

$f(0) = 4$

$f(2) = 0$

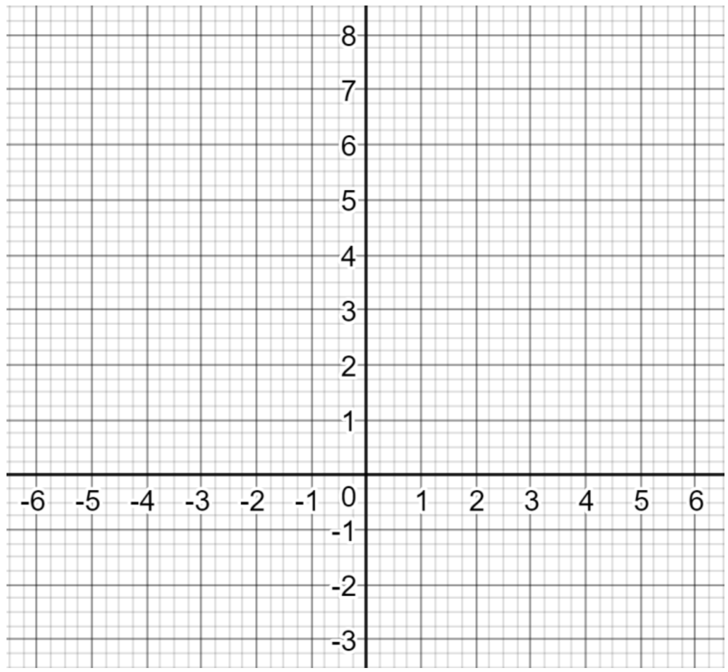
$f''(x) > 0$, for $x > 0$

$f'(2) = f'(-2) = 0$

$f'(x) < 0$ for $|x| < 2$

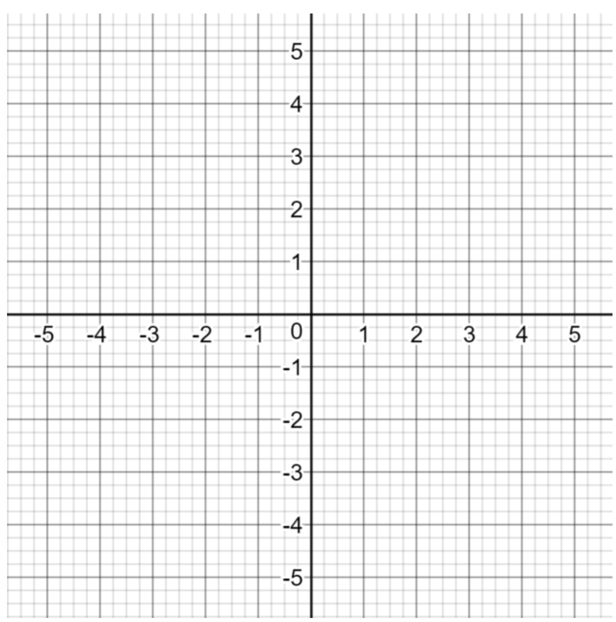
$f''(x) < 0$ for $x < 0$

$f'(x) > 0$ for $|x| > 2$



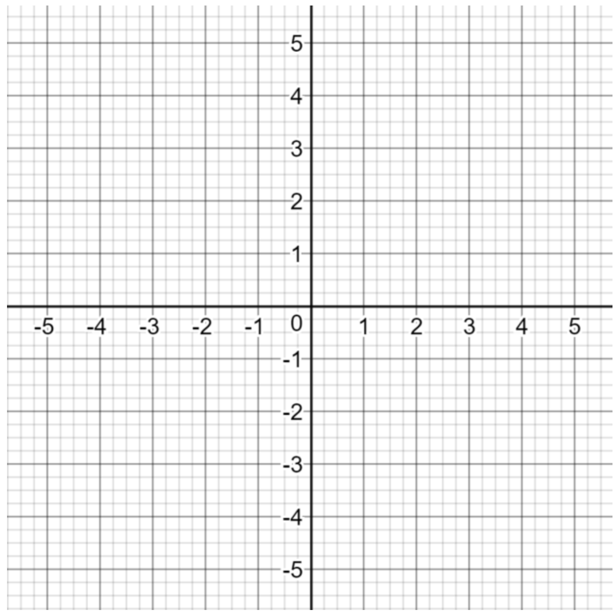
Example 4 Sketch a curve illustrating a function such that:

- It is symmetrical across the y-axis and $f(0) = 2$
- It has a horizontal asymptote: $y = 0$ and two vertical asymptotes: $x = \pm 2$
- It is increasing on $(0, 2)$ and decreasing on $(-\infty, -2)$ and $(-2, 0)$
- It is concave up on $(-2, 2)$ and concave down on $(-\infty, -2)$ and $(2, \infty)$



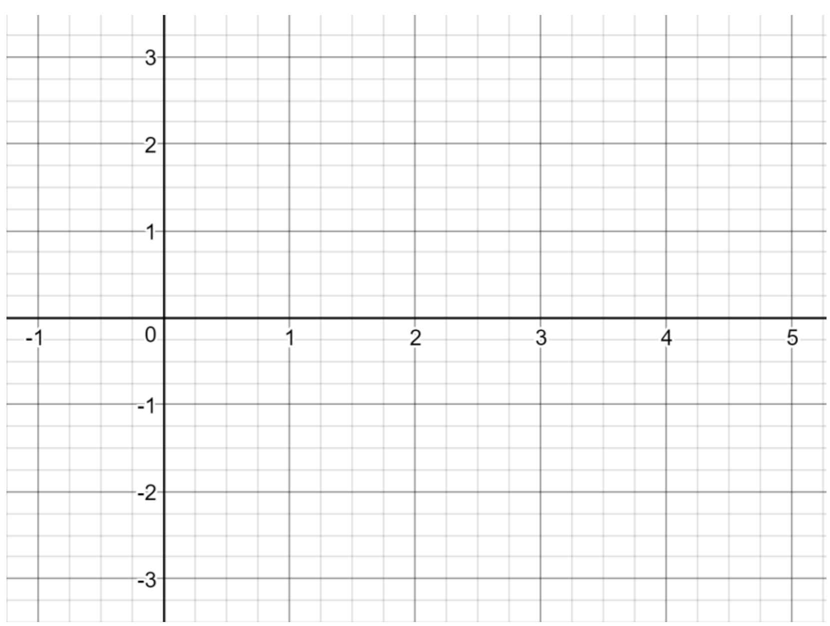
Example 5 Sketch a curve illustrating a function such that:

- It is increasing on $(-\infty, 0)$ and $(1, \infty)$ and decreasing on $(0, 1)$
- It has a tangent with undefined slope at the origin
- It has a horizontal tangent at $(1, -1)$
- It is concave up for all x except $x = 0$



Example 6 Let f be a function that is continuous on the interval $[0, 4]$. The function f is twice differentiable except at $x = 2$. The function f and its derivatives have the properties indicated in the table below, where DNE indicates that the derivatives of f do not exist at $x = 2$. Sketch a graph of f.

x	0	$0<x<1$	1	$1<x<2$	2	$2<x<3$	3	$3<x<4$
$f(x)$	-1	Negative	0	Positive	2	Positive	0	Negative
$f'(x)$	4	Positive	0	Positive	DNE	Negative	-3	Negative
$f''(x)$	-2	Negative	0	Positive	DNE	Negative	0	Positive



AB Calculus: Modeling and Optimization   Name: ______________________________

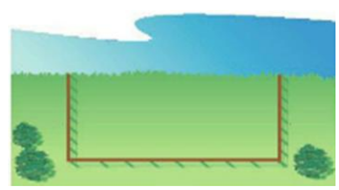
Optimization is a useful application of differential calculus. Any time you use the superlative case – biggest, smallest, cheapest, strongest, prettiest, etc. – you are trying to optimize. What is keeping us from making something infinitely big, infinitely small, and infinitely pretty? Well, there is always some limiting factor, a constraint that prevents that from happening.

When solving an optimization problem, use the following steps:

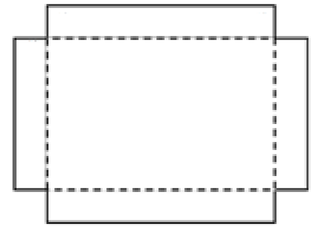
1. Read the problem carefully
2. Draw a clearly labeled picture (if needed) and identify variables
3. State what you are trying to optimize and write an optimization equation for the quantity to be optimized.
4. State any constraints and write a secondary equation involving the constraints if necessary.
5. Solve the secondary equation for any convenient variable and plug it into the optimization equation to establish the equation for the optimal quantity in terms of a single variable.
6. Simplify the optimization equation.
7. Take the derivative of the optimization equation and find critical points. You will also need to consider endpoints, if they exist.
8. Confirm that the critical point/endpoint is a max or min using the first or second derivative test. This step is important and cannot be skipped.
9. Answer the question in a complete sentence with appropriate units.
10. There really is not a step 10, I just wanted to have a 10 since it was so close ☺

Example 1 Two positive numbers have a sum of 60. What is the maximum product of one number times the square of the second number?

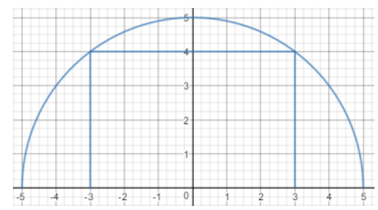
Example 2 You have 40 feet of fence to enclose a rectangular garden along the side of a lake. What is the maximum area that you can enclose?



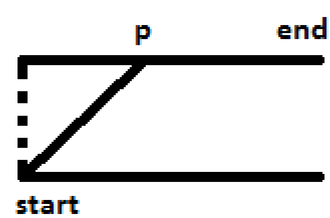
Example 3 An open-top box is made by cutting congruent squares of side length x from the corners of a 20-by-25 inch sheet of tin and bending up the sides. How large should the squares be to make the box hold as much as possible?



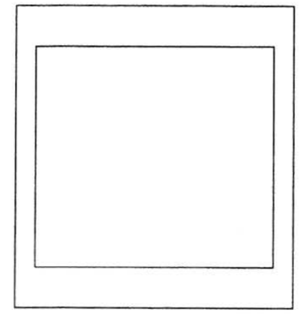
Example 4 A rectangle is bounded by the x-axis and the semicircle $y = \sqrt{25 - x^2}$. Find the dimensions of the rectangle that would maximize the area of the rectangle.



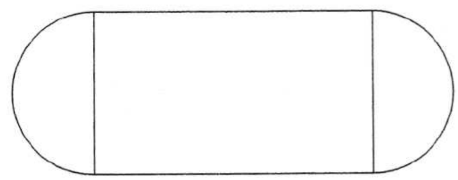
Example 5 In a swim and run biathlon, Jill must get to a point on the other side of a 50 meter wide river, 100m downstream from her starting point. She can swim at a rate of 2 m/sec and run at a rate of 5 m/sec. Toward what point on the opposite side of the river should Jill swim in order to minimize her total time?



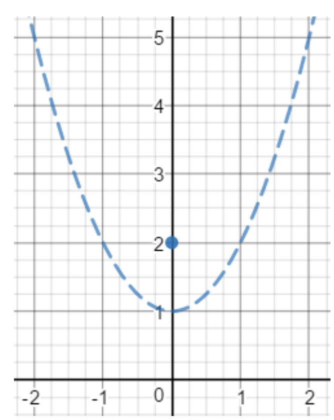
Example 6 A rectangular page is to contain 24 square inches of print. The margins at the top and bottom of the page are to be 1.5 inches, and the margins on the left and right are to be 1 inch. What should the dimensions of the page be so that the least amount of paper is used?



Example 7 An indoor physical fitness room consists of a rectangular region with a semicircle on each end. The perimeter of the room is to be a 200-meter running track. Find the dimensions of the rectangular region of the track that will maximize the area of the rectangular region.



Example 8 Determine the point(s) on $y = x^2 + 1$ that are closest to $(0, 2)$.



Example 9 A 2 feet piece of wire is cut into two pieces and one piece is bent into a square and the other is bent into an equilateral triangle. Where should the wire be cut so that the total area enclosed by both is a minimum (Note: it is not required to build both shapes)

Example 10 The manager of an 80 unit apartment complex is trying to decide what rent to charge. Experience has shown that at a rent of \$600, all of the units will be full. On the average, one additional unit will remain vacant for each \$20 increase in rent. Find the rent to charge so as to maximize revenue.