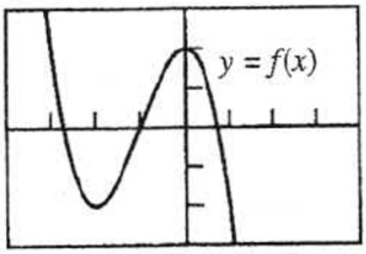
ABCALC LHop and Related Rates Review Solutions

1. Use the graph of $f(x)$ shown to the right to estimate the answer to the following questions.



$[-4, 4]$ by $[-3, 3]$

a) Find the local extreme values and where they occur.

Min: -2 at $x=-2$ max: 2 at $x=0$

b) For what values of x is $f'(x) > 0$?

$(-2,0)$

c) For what values of x is $f'(x) < 0$?

$(-4,-2) \cup (0,4)$

d) For what values of x is $f''(x) > 0$?

$(-4,-1)$

e) For what values of x is $f''(x) < 0$?

$(-1,4)$

Nov 13-10:19 PM

2. For $f(x) = -x^4 + 18x^2 + 11$, find the following without using a calculator.

$f' = -4x^3 + 36x$

$0 = -4x^3 + 36x$

$0 = -4x(x^2 - 9)$

$0 = -4x(x-3)(x+3)$

$x = 0,\ x = 3,\ x = -3$

f' + − + −

f ↗ -3 ↘ 0 ↗ 3 ↘

a) Intervals over which $f(x)$ is increasing. Justify your response.

$(-\infty,-3) \cup (0,3)$ $f' > 0$

b) Intervals over which $f(x)$ is decreasing. Justify your response.

$(-3,0) \cup (3,\infty)$ $f' < 0$

c) Intervals over which $f(x)$ is concave up. Justify your response.

$(-\sqrt{3}, \sqrt{3})$ $f'' > 0$

d) Intervals over which $f(x)$ is concave down. Justify your response.

$(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$ $f'' < 0$

e) x values of all relative extreme values. Label each as a relative max or min. Justify your response.

max $x=3, x=-3$ f' changes + to −   min $x=0$ f' changes − to +

f) x values of all points of inflection. Justify your response.

$x = -\sqrt{3}$ and $x = \sqrt{3}$ f'' changes sign

$f'' = -12x^2 + 36$

$0 = -12(x^2 - 3)$

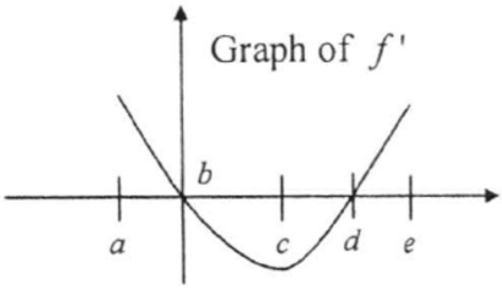
$x = \sqrt{3} \quad x = -\sqrt{3}$

f'' − + −

f ∩ $-\sqrt{3}$ ∪ $\sqrt{3}$ ∩

Nov 13-10:22 PM

3. The graph of $f'(x)$, the derivative of f, is shown to the right. Use it to find the following.



a) Intervals over which $f(x)$ is increasing. Justify your response.

$(a,b) \cup (d,e)$ $\quad f'>0$

b) Intervals over which $f(x)$ is decreasing. Justify your response. (b,d) $\quad f'<0$

c) Intervals over which $f(x)$ is concave up. Justify your response. (c,e) $\quad f''>0$

d) Intervals over which $f(x)$ is concave down. Justify your response. (a,c) $\quad f''<0$

e) x-values where f has a relative extreme. Label each as a max or min. Justify your response.

max $x=b$ $\quad f'$ changes + to −

min $x=d$ $\quad f'$ changes − to +

f) x-values where f has a point of inflection. Justify your response.

$x=c$ $\quad f''$ changes sign

Nov 13-10:26 PM

4. Determine whether the function $f(x) = -2x^2 + 16x - 3$ satisfies the hypothesis of the MVT over the interval $[-2, 3]$. If it does, find the value of x guaranteed by the theorem.

f(x) continuous over [−2,3] also differentiable, MVT applies

$f'(x) = -4x+16 \qquad \frac{f(3)-f(-2)}{3--2} = -18+48-3-(-8-32-3) = \frac{27--43}{5} = \frac{70}{5} = 14$

$-4x+16 = 14$

$-4x = -2$

$\boxed{x = \frac{1}{2}}$

Nov 13-10:28 PM

5. Find the absolute extrema for the function $f(x) = 2x^3 + 3x^2 - 12x + 4$ over the interval $[-4, 2]$.

$f'(x) = 6x^2 + 6x - 12$ never und.

$0 = 6(x^2 + x - 2)$

$0 = 6(x+2)(x-1)$

$x = 1 \quad x = -2$

x	$f(x)$
-4	-28
-2	24
1	-3
2	8

Absolute max 24 at $x = -2$

Absolute min -28 at $x = -4$

Nov 13-10:32 PM

6. Use a linearization to approximate $\sqrt{99.9}$.

$f(x) = \sqrt{x}$ point: $(100, 10)$

$f'(x) = \frac{1}{2\sqrt{x}}$ $f'(100) = \frac{1}{2 \cdot 10} = \frac{1}{20}$

$y - 10 = \frac{1}{20}(x - 100)$

$L(x) = \frac{1}{20}(x - 100) + 10$

$L(99.9) = 0.05(-.1) + 10 = -.005 + 10 = \boxed{9.995}$

Nov 13-10:35 PM

7 Evaluate each of the following limits.

a) $\lim_{x\to 0^+} 5x^2 \ln x$

$$\lim_{x\to 0^+} \frac{5\ln x}{\frac{1}{x^2}} = \frac{-\infty}{\infty}$$

$$\lim_{x\to 0^+} \frac{\frac{5}{x}}{\frac{-2}{x^3}} \cdot \frac{x^3}{-2}$$

$$\lim_{x\to 0^+} \frac{5x^2}{-2} = \boxed{0}$$

b) $\lim_{x\to\infty} 4xe^{-x}$

$$\lim_{x\to\infty} \frac{4x}{e^x} = \boxed{0}$$

Nov 14-1:27 PM

c) $\lim_{x\to\frac{\pi}{2}} (3\sec x - 3\tan x)$ und − und

$$\lim_{x\to\frac{\pi}{2}} \frac{3}{\cos x} - \frac{3\sin x}{\cos x}$$

$$\lim_{x\to\frac{\pi}{2}} \frac{3-3\sin x}{\cos x} = \frac{0}{0}$$

$$\lim_{x\to\frac{\pi}{2}} \frac{-3\cos x}{-\sin x} = \frac{0}{-1} = \boxed{0}$$

d) $\lim_{x\to\infty} \left(\frac{x^2}{x-1} - \frac{x^2}{x+1}\right)$ $\infty - \infty$

$$\lim_{x\to\infty} \frac{x^2(x+1) - x^2(x-1)}{(x-1)(x+1)}$$

$$\lim_{x\to\infty} \frac{x^3+x^2-x^3+x^2}{x^2-1}$$

$$\lim_{x\to\infty} \frac{2x^2}{x^2-1} = \boxed{2}$$

Nov 14-1:29 PM

e) $\lim_{x\to 0^+} 5(\tan x)^{\sin x}$ $5 \lim_{x\to 0^+} \tan x^{\sin x}$

- $\lim_{x\to 0^+} \sin x \ln(\tan x)$
- $\lim_{x\to 0^+} \frac{\ln(\tan x)}{\csc x} = \frac{\infty}{\infty}$
- $\lim_{x\to 0^+} \frac{\frac{1}{\tan x}\cdot \sec^2 x}{-\csc x \cot x}$
- $\lim_{x\to 0^+} \frac{\cos x}{\sin x}\cdot\frac{1}{\cos^2 x}\cdot\frac{-\sin x}{1}\cdot\frac{\sin x}{\cos x}$
- $\lim_{x\to 0^+} \frac{-\sin x}{\cos^2 x} = 0$

undo the ln

$e^0 = 1$

from moving the 5 out

$5\cdot 1 = \boxed{5}$

f) $\lim_{x\to\infty} 3x^{\frac{1}{x}}$ $3\lim_{x\to\infty} x^{\frac{1}{x}}$

$\lim_{x\to\infty} \frac{\ln x}{x} = \frac{\infty}{\infty}$

$\lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$

undo the ln

$e^0 = 1$

from moving 3 out front

$3\cdot 1 = \boxed{3}$

Nov 14-1:33 PM

g) $\lim_{x\to 0}\frac{e^x - e^{-x}}{x} = \frac{0}{0}$

$\lim_{x\to 0} \frac{e^x + e^{-x}}{1} = 1+1 = \boxed{2}$

h) $\lim_{x\to 5}\frac{\sqrt{x-4}-1}{x-5} = \frac{0}{0}$

$\lim_{x\to 5} \frac{\frac{1}{2\sqrt{x-4}}(1)}{1} = \frac{1}{2\sqrt{5-4}} = \boxed{\frac{1}{2}}$

Nov 14-1:33 PM

8 Find the differential form of the derivative of $y = e^x \cos^{-1}(x^2)$.

$$\frac{dy}{dx} = e^x \cdot \frac{-1}{\sqrt{1-(x^2)^2}}(2x) + \cos^{-1}(x^2) \cdot e^x$$

$$\boxed{dy = \left(\frac{-2xe^x}{\sqrt{1-x^4}} + e^x \cos^{-1} x^2\right) dx}$$

Nov 14-1:34 PM

9. Suppose the revenue, in dollars, for producing x bicycles is given by $r(x) = 90x$ and the cost to produce the bicycles is given by $c(x) = 0.0002x^3 - 0.1x^2 + 20x + 6000$. Find the production level that will maximize profit.

$P(x) = R(x) - C(x)$

$P(x) = 90x - (.0002x^3 - 0.1x^2 + 20x + 6000)$

$P'(x) = 90 - .0006x^2 + .2x - 20$

$0 = 70 - .0006x^2 + .2x$

$x = X \approx 546.725$

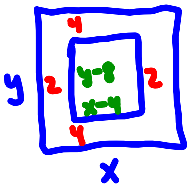
P': $+$ | $-$ max since

P: $\nearrow$ 546.725 $\searrow$ P' changes $+$ to $-$

The company should produce 547 bicycles to maximize profit.

Nov 13-10:40 PM

10. A handbill is to contain 50 square inches, with 4 inch margins at the top and bottom and 2 inch margins on each side. What dimensions for the handbill would give the largest printed area?



$xy = 50 \qquad y = \frac{50}{x}$

$A = (x-4)(y-8)$

$A = (x-4)\left(\frac{50}{x}-8\right)$

$A = 50 - 8x - \frac{200}{x} + 32$

$A' = -8 + \frac{200}{x^2}$

$0 = -8 + \frac{200}{x^2}$

$\frac{200}{x^2} = 8$

$200 = 8x^2$

$25 = x^2$

$x = \pm 5$

$x = 5$

A': $+$ | $-$

A: $\nearrow$ 5 $\searrow$

max since A' changes $+$ to $-$

$y = \frac{50}{5} = 10$

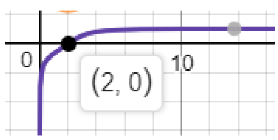
$\boxed{\text{5 in} \times \text{10 in will maximize printed area}}$

11. Find the point on the curve $y = \sqrt{2x}$ that is nearest to the point $(1, 4)$.

minimize distance

$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

$d = \sqrt{(x-1)^2 + (\sqrt{2x}-4)^2}$



$d' = 0$ when $x = 2$

d': $-$ | $+$

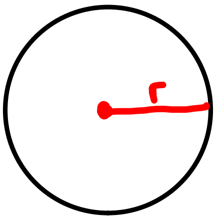
d: $\searrow$ 2 $\nearrow$

min since d' changes $-$ to $+$

when $x = 2$ $\quad y = \sqrt{2 \cdot 2} = 2$

$\boxed{\text{point nearest to } \sqrt{2x} \text{ is } (2,2)}$

12. A thin circular metal disk changes size, but not shape, when heated. The disk is being heated so that its radius is increasing at a rate of 0.03 mm/sec. How fast is the area of the disk changing when the radius is 200 mm?



$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$

$\frac{dA}{dt} = 2\pi(200)(.03)$

$\frac{dA}{dt} = 12\pi \text{ mm}^2/\text{sec}$

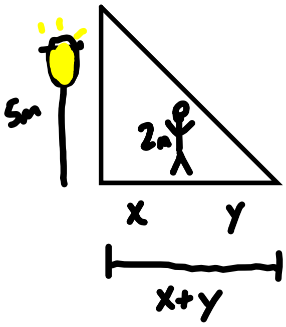
The area is increasing at a rate of 12π mm²/sec when $r = 200$mm

$\frac{dr}{dt} = .03$ mm/sec

find $\frac{dA}{dt}$ when $r = 200$mm

Nov 14-7:43 AM

13 A person 2 m tall walks towards a lamppost on level ground at a rate of 0.5 m/sec. The lamppost is 5 m high. How fast is the length of the person's shadow changing when the person is 3 m from the lamppost?



$\frac{5}{2} = \frac{x+y}{y}$

$5y = 2x + 2y$

$3y = 2x$

$3\frac{dy}{dt} = 2\frac{dx}{dt}$

$3\frac{dy}{dt} = 2(-.5)$

$\frac{dy}{dt} = \frac{-1}{3}$ m/sec

The length of his shadow is changing at a rate of $\frac{-1}{3}$ m/sec when the person is 3 m from the lamppost.

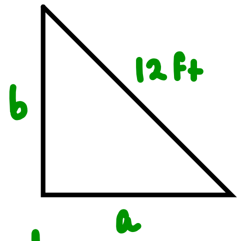
$\frac{dx}{dt} = -.5$ m/sec

Find $\frac{dy}{dt}$ when $x = 3$

Nov 14-7:54 AM

14 An 12 foot ladder stands against a vertical wall. If the lower end of the ladder is being pulled away from the wall at a rate of 2 ft./sec,

a) How fast is the top of the ladder coming down the wall at the instant it is 6 ft. above the ground?



$\frac{da}{dt} = 2$ ft/sec

Find $\frac{db}{dt}$ when $b=6$

$a^2+b^2=12^2$

$2a\frac{da}{dt}+2b\frac{db}{dt}=0$

At instant $b=6$

$12^2=6^2+a^2$

$a=\sqrt{108}$

$2(\sqrt{108})(2)+2(6)\frac{db}{dt}=0$

$4\sqrt{108}+12\frac{db}{dt}=0$

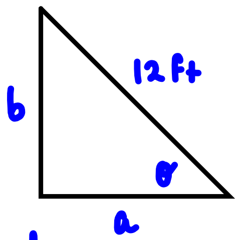
$\frac{db}{dt}=\frac{-4\sqrt{108}}{12} \approx -3.464$ ft/sec

The ladder is moving down the wall at a rate of 3.464 ft/sec when the ladder is 6ft above the ground.

Nov 14-7:58 AM

14 An 12 foot ladder stands against a vertical wall. If the lower end of the ladder is being pulled away from the wall at a rate of 2 ft./sec,

b) How fast is the angle of elevation changing at the instant the ladder is 6 ft. above the ground?



$\frac{da}{dt} = 2$ ft/sec

Find $\frac{d\theta}{dt}$ when $b=6$

$\cos\theta = \frac{adj}{hyp}$ $\sin\theta = \frac{opp}{hyp}$

$\cos\theta = \frac{a}{12}$

$-\sin\theta \cdot \frac{d\theta}{dt} = \frac{1}{12}\cdot\frac{da}{dt}$

$-\frac{6}{12}\cdot\frac{d\theta}{dt} = \frac{1}{12}\cdot 2$

$-\frac{1}{2}\frac{d\theta}{dt} = \frac{1}{6}$

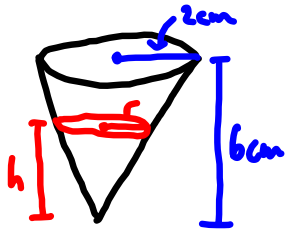
$\frac{d\theta}{dt} = \frac{-1}{3}$ rad/sec

The angle of elevation is decreasing at a rate of 1/3 rad/sec when $b=6$.

Nov 14-7:59 AM

15 A conical cup is 4 cm across and 6 cm deep. Water leaks out of the bottom at a rate of 2 cm^3/sec. How fast is the water level dropping when the height of the water is 3 cm?

$V=\frac{1}{3}\pi r^2 h$



$\frac{dV}{dt} = -2\ cm^3/sec$

Find $\frac{dh}{dt}$ when $h=3$

$\frac{r}{h}=\frac{2}{6}$

$r=\frac{1}{3}h$

$V=\frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h$

$V=\frac{\pi}{27}h^3$

$\frac{dV}{dt}=\frac{\pi}{9}h^2\frac{dh}{dt}$

$-2=\frac{\pi}{9}(3)^2\cdot\frac{dh}{dt}$

$-2=\pi\frac{dh}{dt}$

$\frac{dh}{dt}=\frac{-2}{\pi}$ cm/sec

The height of the water is decreasing at a rate of $\frac{-2}{\pi}$ cm/sec when h=3cm.

Nov 14-8:17 AM

16 If $f(x)=4x^2-4x-11$, find the linearization at $x=-3$, use it to approximate $f(-3.002)$, and use your calculator to determine how accurate the approximation is.

$f(-3)=36+12-11=37$

$f'(x)=8x-4$

$f'(-3)=-28$

$y-37=-28(x+3)$

$L(x)=-28(x+3)+37$

$L(-3.002)=-28(-.002)+37=37.056$

$f(-3.002)\approx 37.056016$

$|Error|=.000016$

$|Error|<10^{-4}$

Nov 14-1:33 PM

17 Let $y = \ln(6x + 7)$. Find dy and estimate dy when $x = 8$ and $dx = 0.04$.

$$\frac{dy}{dx} = \frac{1}{6x+7} \cdot 6$$

$$dy = \left(\frac{6}{6x+7}\right) dx$$

$$dy = \left(\frac{6}{48+7}\right)(.04)$$

$$dy \approx .004$$

Nov 14-1:34 PM