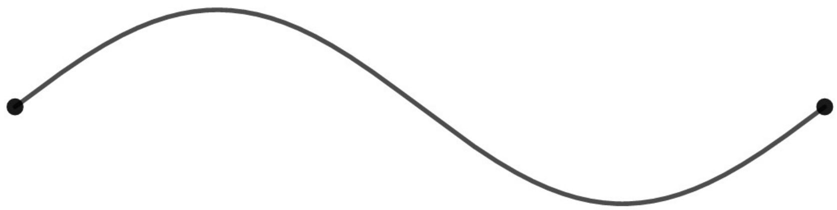
AB Calculus Arc Length Notesheet Name: ___________________________________

We have already seen how definite integrals can be used to find the area under a curve, area between two curves, and to find the volume of a solid of revolution. Definite integrals can also be used to find the arc length of a curve. To approximate the length of a section of the curve, we can use line segments. The length of these segments can be found using the distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, over the length of the interval. An increase in the number of line segments used improves the accuracy of the approximation (See where this is going?). If we find the limit as the number of line segments approaches infinity (through integration), then we can find the exact length of the curve.

$4\sqrt{2}$

$= \sqrt{16 \cdot 2}$



Derive the formula for finding the length of a curve over the interval $[a, b]$.

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$d = \sqrt{dx^2 + dy^2} \cdot \frac{dx}{dx}$

$d = \sqrt{(dx^2 + dy^2)\left(\frac{1}{dx^2}\right)} \cdot dx$

$d = \sqrt{\frac{dx^2}{dx^2} + \frac{dy^2}{dx^2}} \cdot dx$

$d = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$

↳ a single segment

and $\int_a^b \sqrt{1 + (f'(x))^2}\, dx$ adds them all up!

Arc Length of a Curve from $[a, b]$

Let the function $y = f(x)$ represent a smooth curve on the interval $[a, b]$. The arc length of $f(x)$ between a and b is

$$s = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx$$

Similarly, for a smooth curve $x = f(y)$, the arc length of $g(x)$ between c and d is

$$s = \int_c^d \sqrt{1 + \left(g'(y)\right)^2}\, dy$$

When finding the derivative of the function, take the derivative with respect to the variable your function is in terms. In other words, treat the variable that is solved for as "y" and the variable in the function as "x." Here are a few examples:

Function	**Derivative**
$f(x) = 3x^2$	$f'(x) = 6x$
$g(y) = 2y^4 + 3y$	$g'(y) = 8y^3 + 3$
$s(t) = \sin t$	$s'(t) = \cos t$

Example 1 Find the length of the curve $y = \frac{x^3}{6} + \frac{1}{2x}$ over the interval $\left[\frac{1}{2},\ 2\right]$.

$$y = \frac{1}{6}x^3 + \frac{1}{2}x^{-1}$$

$$L = \int_{1/2}^{2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx = 2\,0625$$

$$\frac{dy}{dx} = \frac{1}{2}x^2 - \frac{1}{2}x^{-2}$$

Example 2 Find the length of the curve $y = \frac{2}{3}x^{\frac{3}{2}} + 1$ over the interval $[0, 1]$.

$$\frac{dy}{dx} = x^{\frac{1}{2}}$$

$$L = \int_0^1 \sqrt{1 + \left(x^{\frac{1}{2}}\right)^2}\ dx \approx 1\,219$$

practice doing by hand

$$\int_0^1 \sqrt{1+x}\ dx = \frac{2}{3}(1+x)^{\frac{3}{2}}\Big|_0^1 = \frac{2}{3}(1+1)^{\frac{3}{2}} - \frac{2}{3}(1+0)^{\frac{3}{2}} = \frac{2}{3}(2)^{\frac{3}{2}} - \frac{2}{3}(1)^{\frac{3}{2}}$$

$$\boxed{= \frac{2}{3}\left(\sqrt{8} - 1\right)}$$

Example 3 Find the length of the curve $x = e^{-y}$ from $y = 0$ to $y = 2$.

$$\frac{dx}{dy} = -e^{-y}$$

$$L = \int_0^2 \sqrt{1 + \left(-e^{-y}\right)^2}\ dy \approx 2\,221$$

Example 4 Find the length of the curve $x = \sqrt{36 - y^2}$ from $y = 0$ to $y = 3$.

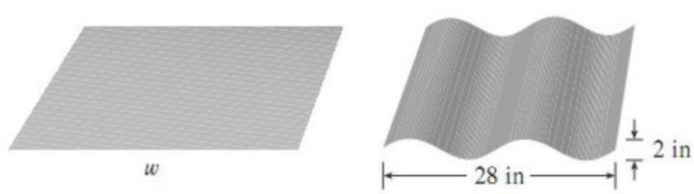
$$x = (36 - y^2)^{\frac{1}{2}}$$

$$\frac{dx}{dy} = \frac{1}{2}(36 - y^2)^{-\frac{1}{2}} \cdot (-2y) = \frac{-y}{\sqrt{36 - y^2}}$$

$$L = \int_0^3 \sqrt{1 + \left(\frac{-y}{\sqrt{36 - y^2}}\right)^2}\ dy \approx 3\,142$$

Example 5 A manufacturer of corrugated panel roofing wants to produce "C" panels that are 28 inches wide and 2 inches thick by processing flat sheets of metal as shown in the figure below. The profile of the roofing takes the shape of the sine wave $y = \sin\left(\frac{\pi x}{7}\right)$. Find the width w of a flat metal sheet that is needed to make a 28 inch panel. Assume the process does not stretch the material. Round to 3 decimal places if necessary.

$$y = \sin\left(\frac{\pi}{7}x\right)$$



$$\frac{dy}{dx} = \frac{\pi}{7}\cos\left(\frac{\pi}{7}x\right)$$

$$\int_0^{28} \sqrt{1 + \left(\frac{\pi}{7}\cos\left(\frac{\pi}{7}x\right)\right)^2}\ dx \approx 29\,361$$