$y(0)=10 \quad y(1)=11 \quad y(2)=10 \quad y(3)=7 \quad y(4)=2$

Δx

$\frac{4-0}{4}=1$

AB Calculus Definite Integrals and Antiderivatives Notesheet **Name:** ______________________________

1. Approximate the area under $y=-x^2+2x+10$ over [0, 4] using 4 subintervals and the given method.

a) LRAM $1(10+11+10+7)=38$

Trap $\frac{1}{2}(b_1+b_2)h$

b) RRAM $1(11+10+7+2)=30$

c) MRAM $1(10.75+10.75+8.75+4.75)=35$

d) Trapezoid Rule $\frac{1}{2}(1)\left(10+2(11+10+7)+2\right)=34$

2. Evaluate the following integrals.

a) $\int_{-4}^{-1} \frac{\pi}{2}\,d\theta$ $\quad \frac{3\pi}{2}$

b) $\int_0^4 \sqrt{16-x^2}\,dx$ $\quad A=\frac{1}{4}\pi r^2 = \frac{1}{4}\pi(4)^2 = 4\pi$

3. Multiple Choice: If n is a positive integer, then $\lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\cdots+\left(\frac{3n}{n}\right)^2\right]$ can be expressed as

A) $\int_0^1 \frac{1}{x^2}\,dx$ B) $\int_0^1 3\left(\frac{1}{x}\right)^2 dx$ C) $\int_0^3 \left(\frac{1}{x}\right)^2 dx$ D) $\int_0^3 x^2\,dx$ E) $3\int_0^1 x^2\,dx$

AROC

$f(3)=12$

$f(1)=0$

4. Let f be the function given by $f(x)=x^3-7x+6$. Find the number c that satisfies the conclusion of the Mean Value Theorem of Derivatives for f on the closed interval [1, 3].

$\frac{12-0}{3-1}=\frac{12}{2}=6$ / IROC $f'(x)=3x^2-7$ / $3x^2-7=6$, $3x^2=13$, $x^2=\frac{13}{3}$, $x=\sqrt{13/3}$

In this last section we defined the definite integral as a limit of a Riemann Sum, thus we can use the properties of limits to develop properties of the definite integral. The proofs of each of the rules below are derived directly from the properties of limits and Riemann Sums.

Rules for Definite Integrals			
#	**Rule**	**Notation**	**Statement**
1.	Order of Integration	$\int_a^b f(x)dx = -\int_b^a f(x)dx$	If you reverse the order of integration, you get the opposite answer.
2.	Zero	$\int_a^a f(x)dx = 0$	This should make sense if you think about the area of a rectangle with no width.
3.	Constant Multiple coefficient	$\int_a^b k\cdot f(x)dx = k\cdot\int_a^b f(x)dx$	Taking the constant out of the integral many times makes it simpler to integrate.
4.	Sum and Difference	$\int_a^b [f(x)\pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$	This allows you to integrate functions that are added or subtracted separately.
5.	Additivity transitive	$\int_a^b f(x)dx + \int_b^c f(x)dx = \boxed{\int_a^c f(x)dx}$	Pay close attention to the limits of integration … this comes in handy when dealing with total area or other functions where we need to break them into smaller parts.

Note: Notice how there are no rules here for two functions that are multiplied or divided … that comes later!

Example 1: Given $\int_2^6 f(x)dx = 10$ and $\int_2^6 g(x)dx = -2$, find the following:

a) $\int_2^6 [f(x) + g(x)]dx = \int_2^6 f(x)dx + \int_2^6 g(x)dx$

$= 10 + (-2) = 8$

b) $\int_2^6 [g(x) - f(x)]dx = \int_2^6 g(x)dx - \int_2^6 f(x)dx$

$-2 - 10 = -12$

c) $\int_2^6 3f(x)dx = 3\int_2^6 f(x) = 3(10)$

$= 30$

d) $\int_2^6 (f(x)+2)\, dx =$

$= \int_2^6 f(x)dx + \int_2^6 2dx = 10 + 8 = 18$

Example 2: Given $\int_0^5 f(x)dx = 10$ and $\int_5^7 f(x)dx = 3$, find the following:

a) $\int_0^7 f(x)dx = \int_0^5 f(x)dx + \int_5^7 f(x)dx$

$= 10 + 3 = 13$

b) $\int_5^0 f(x)dx = -\int_0^5 f(x)dx = -10$

c) $\int_5^5 f(x)dx = 0$

d) $\int_0^5 3f(x)dx = 3\int_0^5 f(x)dx$

$= 3(10) = 30$

Average Value of a Function

Suppose you wanted to find the average temperature during a 24 hour period. How could you do it?

Suppose $f(t)$ represents the temperature at time t measured in hours since midnight. One way to start is to measure the temperature at *n* equally spaced times $t_1, t_2, t_3, \ldots\ t_n$ and then average those temperatures.

Example 3: Using this method, write an expression for the average temperature.

The larger the number of measurements, the more accurately this will reflect the average temperature. Notice we can write this expression as a Riemann sum by first noting that the interval between measurements will be $\Delta t = \frac{24}{n}$, so $n = \frac{24}{\Delta t}$.

Example 4: Substitute this value of n into your expression above and simplify.

Example 5: The last expression gives us an approximate Average Temperature. As $n \to \infty$ (meaning we are taking a lot of temperature readings) this Riemann Sum becomes a definite integral. Write the definite integral that gives us the average temperature since midnight.

Example 6: Do you think that there is any point during the day that the temperature reading on the thermometer is the *exact* value of the average temperature?

The process that we just used to find the average temperature is used to find the average value of any function.

The Average Value of a Function

If f is integrable on [a, b], its average value on [a, b] is given by

$\frac{x}{4}$ $\frac{1}{4}x$

$$Average\ Value = \frac{1}{b-a}\int_a^b f(x)dx \text{ or } Average\ Value = \frac{\int_a^b f(x)dx}{b-a}$$

The average value of a function is just the integral over the interval.

To get a more geometric idea of what the average value is, complete the following example.

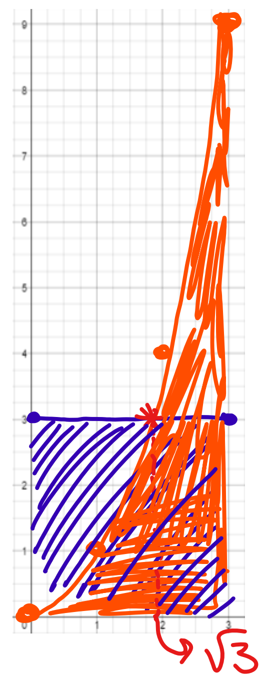
Example 7

a) Graph the function $y = x^2$ on [0, 3] on the grid to the right.

b) Set up a definite integral to find the average value of y on [0, 3], then use your calculator to evaluate the definite integral.

average value of y over [0,3] $= \frac{\int_0^3 x^2 dx}{3-0} = \frac{9}{3} = 3$

c) Graph this value as a function on the grid to the right. Does this function ever actually equal this value? If so, at what point(s) in the interval does the function assume its average value? Graph 3 on the grid →



d) What do you suppose is the relationship between the area the x-axis and the curve $y = x^2$ on [0, 3] and the area of the rectangle formed using the average value as the height and the interval [0, 3] as the width?

they are equal to each other

The Mean Value Theorem for Definite Integrals

The Mean Value Theorem for Integrals basically says that if you are finding the area under a curve between x = a and x = b, then there is some number c between a and b whose function value you can use to form a rectangle that has an area equal to the area under the curve.

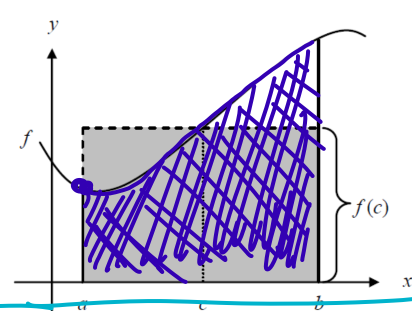
Example 8

a) What is an integral expression that could be used to determine the area under the curve from a to b?

$\int_a^b f(x)dx$

b) What is the area of the shaded rectangle?

$(b-a)f(c)$



This value of $f(c)$ is just the average value of f on the interval [a, b].

$(b-a)f(c) = \int_a^b f(x)dx$ → $f(c) = \frac{\int_a^b f(x)dx}{b-a}$

So another way to look at this is the Mean Value Theorem for integrals just says that at some point within the interval the function MUST equal its average value.

Mean Value Theorem for Integrals

If f is continuous on [a, b], then at some point c in [a, b]

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

Once again, this is an existence theorem. It tells us that c exists, but does not help you find it.

It is greatly important that you understand the difference between average rate of change and average value. For now, understand that the average rate of change is the slope between two points on a given function and the average value is the integral divided by the interval.

Example 9 Find the average value of the function $y = -3x^2 - 1$ over the interval [0, 1].

$$= \frac{\int_0^1 (-3x^2 - 1)\,dx}{1-0} = \frac{\left[-x^3 - x\right]_0^1}{1} = \frac{(-1^3 - 1) - (-0^3 - 0)}{1-0} = \frac{-2}{1} = -2$$

Example 10: (2014 AP Calculus AB Free Response Question #1)

Grass clippings are placed in a bin, where they decompose. For $0 \le t \le 30$, the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t$, where $A(t)$ is measured in pounds and t is measured in days.

a) Find the average rate of change of $A(t)$ over the interval $0 \le t \le 30$. Indicate units of measure.

b) Find the value of $A'(15)$. Using correct units, interpret the meaning of the value in the context of the problem.

c) Find the time t for which the amount grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval $0 \le t \le 30$.

d) For $t > 30$, $L(t)$, the linear approximation to A at $t = 30$, is a better model for the amount of grass clippings remaining in the bin. Use $L(t)$ to predict the time at which there will be 0.5 pounds of grass clippings remaining in the bin. Show the work that leads to your answer.