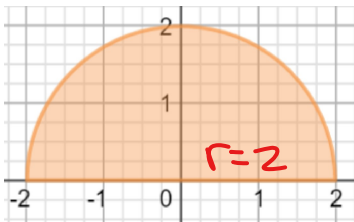


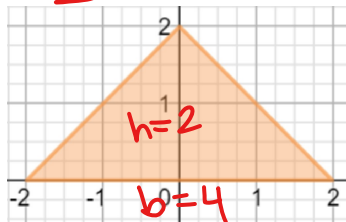
Find the area of the following shapes

$$\frac{\pi r^2}{2}$$



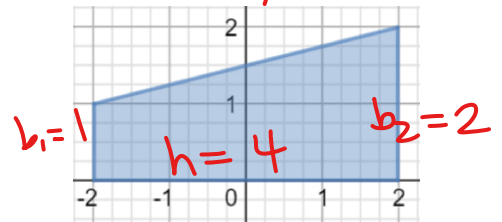
Area:  $\frac{\pi(2)^2}{2} = 2\pi$

$$\frac{1}{2}bh$$



Area:  $\frac{1}{2}(4)(2) = 4$

$$\frac{1}{2}(b_1+b_2)h$$



Area:  $\frac{1}{2}(1+2)(4) = 6$

$$\frac{30-0}{6} = 5$$

Time (milliseconds)	0	5	10	15	20	25	30
Light Output (millions of lumens)	0	0.2	0.6	2.6	4.2	3	1.8

The data in the table represents the rate of light output of a flash bulb at a given time. Find four possible estimates for the total light output of the bulb, measured in million lumen milliseconds. First, find the LRAM, RRAM, and TRAP approximations with 6 equal subintervals. Second, find MRAM using 3 equal subintervals.

53 ← LRAM =  $5(0) + 5(2) + 5(6) + 5(2.6) + 5(4.2) + 5(3)$

62 ← RRAM =  $5(2 + 6 + 2.6 + 4.2 + 3 + 1.8)$

57.5 ← Trapezoid  $\frac{1}{2}(0 + 2(2 + 6 + 2.6 + 4.2 + 3) + 1.8)(5)$

MRAM =  $10(2) + 10(2.6) + 10(3) = 58$

**Introduction**

In the last section we found the area under a curve by finding the area of a finite number of rectangles (LRAM, RRAM, and MRAM) or a finite number of trapezoids (Trapezoid Rule). Every one of these was an example of what is called a Riemann Sum. We are going to stick with rectangles for the time being.

The following steps illustrate what has to happen in order for the sum to be considered a Riemann Sum:

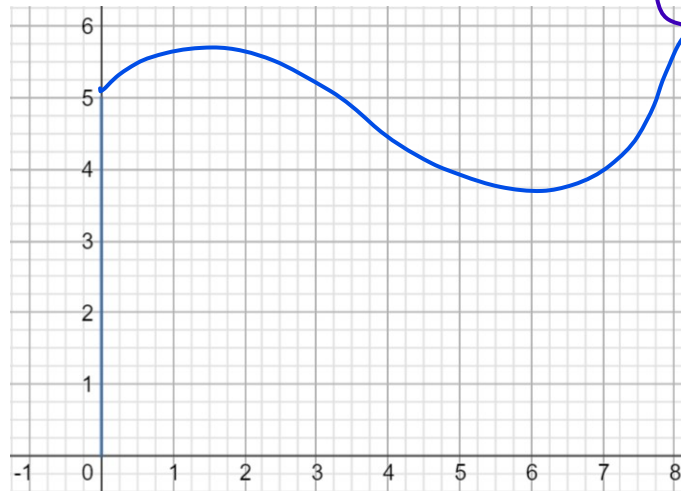
- Step 1:** Start with a continuous function on a closed interval.
- Step 2:** Partition the interval into  $n$  subintervals. The  $k^{\text{th}}$  subinterval has width  $\Delta x_k$ . They can be different.
- Step 3:** In each subinterval, pick any number and call the number picked from the  $k^{\text{th}}$  subinterval  $c_k$  (LRAM picked the left endpoint ... RRAM picked the right endpoint ... MRAM picked the midpoint)
- Step 4:** For each interval, using the width,  $\Delta x_k$ , of the interval as the base, create a rectangle that extends from the x-axis to the function value,  $f(c_k)$ , of the number you picked in each interval. (Note: Some of these rectangles could lie below the x-axis).
- Step 5:** On each interval, form the product  $f(c_k) \cdot \Delta x_k$

Note: If all of our rectangles lie above the x-axis, this would have been the area of each rectangle.

Step 6: Find the sum of each of these products.

$$f(c_1) \cdot \Delta x_1 + f(c_2) \cdot \Delta x_2 + f(c_3) \cdot \Delta x_3 + f(c_4) \cdot \Delta x_4 + \dots + f(c_n) \cdot \Delta x_n = \sum_{k=1}^n f(c_k) \cdot \Delta x_k$$

Handwritten notes: "based on an input" (pointing to  $c_k$ ), "output" (pointing to the sum), "n is an unknown but ideally infinite value" (pointing to  $n$ ), "starting pt" (pointing to  $k=1$ ), "ending point" (pointing to  $n$ ).



$\Sigma \rightarrow$  sigma  
 $\downarrow$   
 Sums up calculations

Following these steps gives you a **Riemann Sum for f on the interval [a, b]**. Every Riemann Sum depends on the partition you choose (i.e. the number of subintervals) and your choice of the number within each interval,  $c_k$ .

### Definite Integral as a Limit of a Riemann Sum

A Definite Integral is defined as a limit of a Riemann Sum.

**Option #1:** If you noticed in step 2 above, we did not care if our subintervals were the same width. If we use the notation  $\|P\|$  to denote the longest subinterval length we can force the longest subinterval length to 0 using a limit of the Riemann Sum as follows:

$$\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(c_k) \cdot \Delta x_k$$

**Option #2:** If we make sure the subintervals are all the same width, we can increase the number of rectangles to infinity using a limit of the Riemann Sum as follows:

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \cdot \Delta x$$

### Notation for Definite Integrals

# of rectangles is approaching  $\infty$

The limit notation we used last is the form we will use to develop Integral notation. As the number of rectangles goes to infinity, the width of each rectangle,  $\Delta x$ , goes to zero. We are going to use the notation  $dx$  to represent this infinitely tiny distance.

The summation notation of sigma is going to be replaced with an Integral Sign,  $\int$ , which looks somewhat like a giant "S" for sum.

$$\Sigma \rightarrow \text{sigma (sum)} = \int \rightarrow \text{integral symbol}$$

$f(x) \rightarrow$  the y-value  $\Delta x \rightarrow dx$

The  $f(c_k)$  which represented a different function value for each interval is going to be replaced with  $f(x)$  since the x-values are going to be soooooo close together it is almost as if we were evaluating the function for every x-value in the interval  $[a, b]$ . Combining all of this we have the following notation:

$\int_a^b f(x) dx \rightarrow$  this is the same as saying  $\lim_{n \rightarrow \infty} \sum_{i=1}^n f(c_i) \Delta x_i$

We read the above notation above as "The Integral of f of x from a to b"

Important Note (Actually, it's a theorem): If the function is continuous, then the definite integral will exist. However, the converse, while true some of the time, is not always true (Calculus BC!)

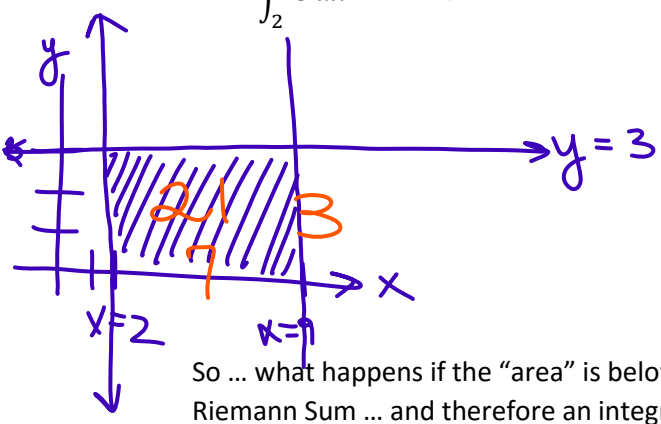
**Using Definite Integrals as Area**

We can define the **area under the curve  $y = f(x)$  from a to b** as an integral from a to b ... As long as the curve is nonnegative and integrable on the closed interval  $[a, b]$ .

Drawing a picture and using geometry is still a valid method of finding areas!

**Example 1:** For each of the following examples, sketch a graph of the function, shade the area you are trying to find, then use geometric formulas to evaluate each integral.

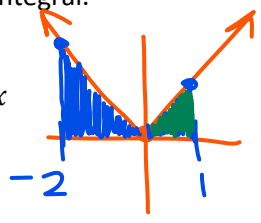
a)  $\int_2^9 3 dx = 21$



b)  $\int_{-2}^1 |x| dx$

$\frac{1}{2}(2)(2) + \frac{1}{2}(1)(1)$

$= 2 + \frac{1}{2} = \frac{5}{2}$

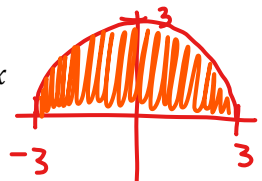


c)  $\int_{-3}^3 \sqrt{9-x^2} dx$

$y = \sqrt{9-x^2} \rightarrow y^2 = 9-x^2$   
 $x^2 + y^2 = 9$

$\frac{\pi r^2}{2} = \frac{\pi(3)^2}{2}$

$= \frac{9\pi}{2}$



So ... what happens if the "area" is below the x-axis ... as I mentioned before, area is inherently positive, but a Riemann Sum ... and therefore an integral can have negative values if the curve lies below the x-axis.

**Example 2:** Consider the function  $f(x) = 3 - x$ . Sketch a graph of this function.

A) What is the area between the curve and the x-axis between  $x = 4$  and  $x = 8$ ?

Area<sub>trap</sub> =  $\frac{1}{2}(1+5)(4) = 12$

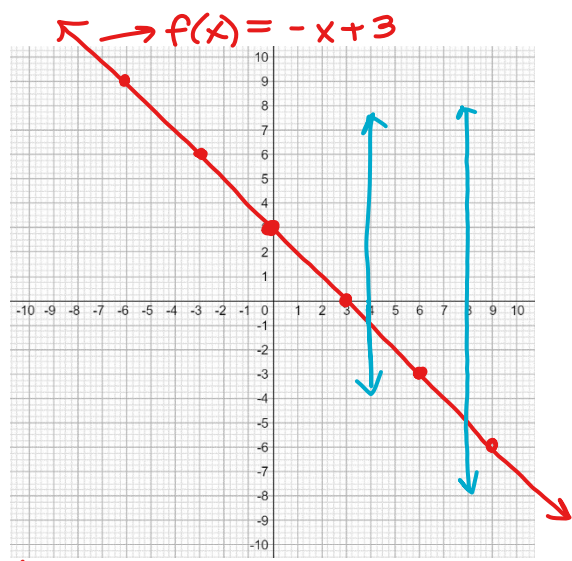
B)  $\int_4^8 (3-x) dx = -12$

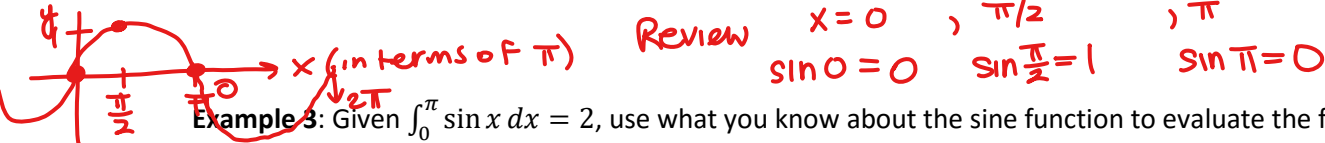
If using calculus only

$\left[ 3x - \frac{1}{2}x^2 \right]_4^8$

$\left[ 3(8) - \frac{1}{2}(8)^2 \right] - \left[ 3(4) - \frac{1}{2}(4)^2 \right] = (24 - 32) - (12 - 8)$

$-8 - 4 = -12$





**Example 3:** Given  $\int_0^\pi \sin x \, dx = 2$ , use what you know about the sine function to evaluate the following integrals.

a)  $\int_\pi^{2\pi} \sin x \, dx = -2$

b)  $\int_0^{2\pi} \sin x \, dx = 0$

c)  $\int_0^{\pi/2} \sin x \, dx = 1$

d)  $\int_{-\pi}^\pi \sin x \, dx = 0$

e)  $\int_0^\pi (2 + \sin x) \, dx = \int_0^\pi 2 \, dx + \int_0^\pi \sin x \, dx$   
 $\boxed{2\pi + 2}$

### The fnInt Function On Your Calculator

By this point, hopefully you understand the following concepts:

1. The limit of a Riemann Sum is used to define a Definite Integral
2. A Definite Integral can be used to find the area under a curve if the curve is above the x-axis. If the curve is below the x-axis, the value of the definite integral is the negative of the value of the area.
3. Since the definite integral can be thought of as area, you can draw a picture and use geometric formulas to find the areas.

BUT ... what happens if you were asked right now, this instant, today to find the definite integral of a function that doesn't lend itself to nice geometric shapes? The good news for now, is that you don't even have to worry about how to do these by hand ... Yet! You get to use your calculator! ... at least for today!

### How to use your calculator to find a definite integral:

On TI-83 Family or TI-84 with old OS	On TI-84 Family with new OS
1. Press MATH 2. Select 9:fnInt( 3. The syntax is as follows:  $\text{fnInt}(\text{function}, x, \text{lower bound}, \text{upper bound})$	1. Press MATH 2. Select 9:fnInt( 3. The syntax is as follows:  $\int_{\text{lower bound}}^{\text{upper bound}} (\text{function}) \, dX$

**Example 4:** Evaluate  $\int_1^3 (x^2 - 2x + 2) \, dx$

**Example 5:** Evaluate  $3 + 2 \int_0^\pi \tan x \, dx$

You can do the same thing from the graphing screen Press 2<sup>nd</sup> Trace (which is CALC), 7:  $\int f(x) \, dx$ , enter lower bound as 1, enter upper bound as 8. The down side to using this method is that you must be able to set your window to see everything.

**Example 6:** Graph  $y = \sqrt{x}$  on a standard viewing window (Zoom: 6). Evaluate  $\int_1^8 \sqrt{x} \, dx$