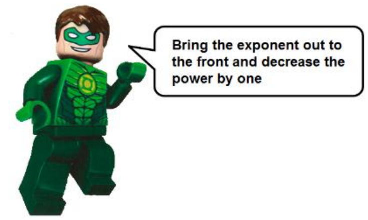
AB Calculus: Rules for Differentiation **Name: ______________________**

Power Rule for Differentiation

If n is any number, then $\frac{d}{dx}[x^n] = n \cdot x^{n-1}$, provided x^{n-1} exists.



The key to using the power rule is to get comfortable using exponent rules to write a function as a power of x.

Example 1: Find the derivative of each of the following.

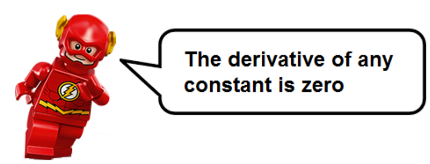
a) $f(x) = x^5$

b) $f(x) = \sqrt[3]{x^2}$

c) $f(x) = \frac{1}{x^4}$

The Derivative of a Constant Function

If c is any constant value, then $\frac{d}{dx}[c] = 0$.

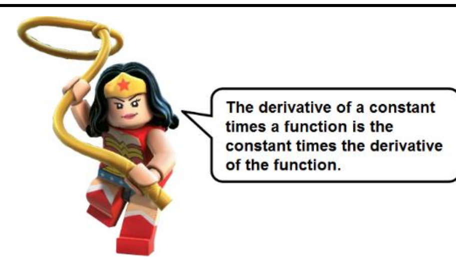


Example 2: Let $f(x) = 5$. Find $f'(x)$.

The Constant Multiple Rule for Derivatives

If u is a differentiable function of x and c is a constant value, then

$$\frac{d}{dx}[cu] = c\frac{du}{dx}$$



Example 3: Find the derivative of each of the following.

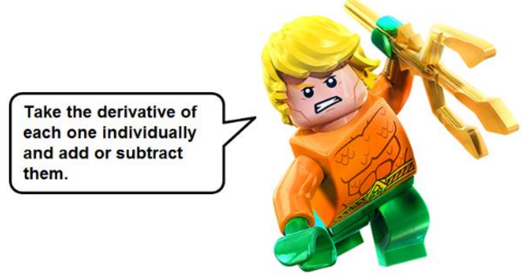
a) $f(x) = 5x^7$

b) $f(x) = \frac{4}{5x^3}$

The Sum and Difference Rule for Derivatives

If u and v are differentiable functions of x, then wherever u and v are differentiable

$$\frac{d}{dx}[u \pm v] = \frac{du}{dx} \pm \frac{dv}{dx}$$



Example 3: Find the derivative of each of the following.

a) $f(x) = x^3 + 4x^2 - 2x + 7$

b) $f(x) = \frac{3}{(-2x)^4} - \frac{x}{2} + \frac{1}{4}$

Example 4: Find the equation of the tangent line to the function $f(x) = 4x^3 - 6x + 5$ when $x = 2$.

Example 5: Let $h(x) = (4x^2 + 1)(2x - 5)$. Find $h'(x)$.

Example 6: The volume of a cube with sides of length s is given by the formula $V = s^3$. Find $\frac{dV}{ds}$ when $s = 4$cm.

We have already seen that the derivative of the sum of two functions is the sum of the derivatives of the two functions. This does not work for the product or quotient of two functions. To illustrate this, look at the following example:

Example 7: $\frac{d}{dx}[x^2 \cdot 3x]$

The Product Rule for Derivatives

If u and v are differentiable functions of x, then

$$\frac{d}{dx}[u \cdot v] = u \cdot \frac{dv}{dx} + v\frac{du}{dx}$$

Example 8: Find $\frac{dy}{dx}$ of each of the following functions.

a) $y = (3 + 2\sqrt{x})(5x^3 - 7)$

b) $y = (3x - 2x^2)(4 + 5x)$

Note: It is also valid to multiply out the function first and then take the derivative.

The Quotient Rule for Derivatives

If u and v are differentiable functions of x, then

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v \cdot \frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example 9: Find the derivative of each of the following.

a) $f(x) = \frac{x}{x^2 + 1}$

b) $f(x) = \frac{5x^2}{x^3 + 1}$

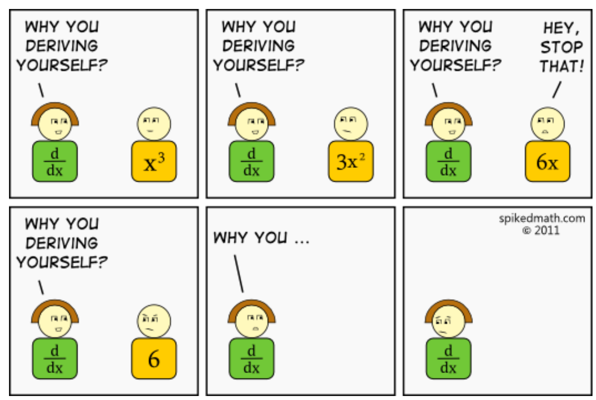
Second and Higher Order Derivatives

The first derivative of y with respect to x is denoted y', $f'(x)$, or $\frac{dy}{dx}$. The second derivative with respect to x is denoted y'', $f''(x)$, or $\frac{d^2y}{dx^2}$. The second derivative is an example of a higher order derivative. We can continue to take derivatives (as long as they exist) using the following notation:

First Derivative	y'	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}[f(x)]$
Second Derivative	y''	$f''(x)$	$\frac{d^2y}{dx^2}$	$\frac{d^2}{dx^2}[f(x)]$
Third Derivative	y'''	$f'''(x)$	$\frac{d^3y}{dx^3}$	$\frac{d^3}{dx^3}[f(x)]$
Fourth Derivative	$y^{(4)}$	$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$	$\frac{d^4}{dx^4}[f(x)]$
Nth derivative	$y^{(n)}$	$f^{(n)}(x)$	$\frac{d^ny}{dx^n}$	$\frac{d^n}{dx^n}[f(x)]$

Example 10: Find the indicated derivative of each of the following.

a) $\frac{d^4}{dx^4}[-5x^8 + 2x^6 - 9x^3 + 32x - 1]$



b) $\frac{d^2}{dx^2}\left[\frac{x}{x-1}\right]$

c) Find $\frac{d^5y}{dx^5}$ if $\frac{d^4y}{dx^4} = 2\sqrt{x}$

AB Calculus: Velocity and Other Rates of Change **Name: ______________________________**

Instantaneous Rates of Change

We have already seen that the instantaneous rate of change at a point is the same as the slope of the tangent line at a point a.k.a the derivative at that point. From now on, unless we use the phrase "average rate of change," we will assume that in calculus the phrase "rate of change" refers to the instantaneous rate of change.

If $f(x)$ represents a quantity, then $f'(x)$ represents the instantaneous rate of change of that quantity. $f(x)$ may describe a particle's position or it's velocity. In fact, $f(x)$ can represent any quantity such as the area of a circle, the temperature outside, the amount of rainfall in a region or the number of people infected with a disease. This is the true power of the derivative: it gives us a way to discuss how fast anything is changing at a single moment in time.

Example 1: An ice cream company knows that the cost, C (in dollars), to produce q quarts of cookie dough ice cream is a function of q, so $C = f(q)$.

a) If $f(200) = 70$, what are the units of the 200? What are the units of the 70? Clearly explain what the statement is telling you.

b) If $f'(200) = 3$, what are the units of the 200? What are the units of the 3? Clearly explain what the statement is telling you.

Example 2: The temperature T, in degrees Fahrenheit, of a cold pizza placed in a hot oven is given by $T = f(t)$, where t is the time in minutes since the pizza was put in the oven.

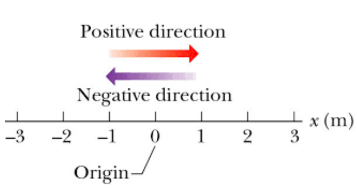
a) What is the meaning of the statement $f(20) = 255$ in this scenario?

b) What is the sign of $f'(t)$? Why?

c) What are the units of $f'(20)$?

d) What is the meaning of the statement $f'(20) = 2$?

Motion along a Line



There is perhaps no more important type of rate of change than that of motion. After all, it was Newton's motivation for inventing calculus in the first place. We have already looked at motion briefly when we investigated the basketball shot from a World Record height. To delve into this topic further, we will need to get familiar with some terminology and relationships between position, velocity, and acceleration.

Relationships between Position, Velocity, and Acceleration

Position functions, often denoted $s(t)$, give the position of an object at any point in time.

The **displacement** of an object is the **total change in position**. If we were given a position function $s(t)$ and an interval of time $[a, b]$, then the displacement would be $s(b) - s(a)$.

The **average velocity** of the object is the **total change in position (displacement) divided by the total change in time**. It can be thought of as the slope of the line connecting two points on a position function.

$$Average\ Velocity = \frac{displacement}{elapsed\ time} = \frac{s(b) - s(a)}{b - a}$$

The **instantaneous velocity** of an object is the derivative of the position function of the object.

$$Instantaneous\ Velocity = v(t) = s'(t) = the\ slope\ of\ the\ tangent\ line\ at\ any\ time\ t$$

Velocity is a quantity that gives both how fast something is moving as well as direction of the movement. If velocity is positive, then movement is occurring in the positive direction. If velocity is negative, then movement is occurring in the negative direction.

Speed is the **absolute value of the velocity**, therefore, speed is always positive.

$$Speed = |v(t)| = |s'(t)|$$

Acceleration is the rate of change of velocity or **the derivative of velocity**. Since it is the derivative of velocity, it is also **the second derivative of position**.

$$Acceleration = v'(t) = s''(t) = the\ slope\ of\ the\ tangent\ line\ to\ the\ velocity\ function\ at\ any\ time\ t$$

The **average acceleration** of an object is the **total change in velocity divided by the total change in time**. It can be thought of as the slope of the line connecting two points on a velocity function.

Example 3: To save his son Luke Skywalker from certain death, Darth Vader picks up The Emperor and throws him off of a ledge on the Death Star. As he falls, The Emperor's position s is given by the function $s(t) = -16t^2 + 16t + 320$, where s is measured in feet and t is measured in seconds.

a) What is The Emperor's displacement from $t = 1$ second to $t = 2$ seconds?

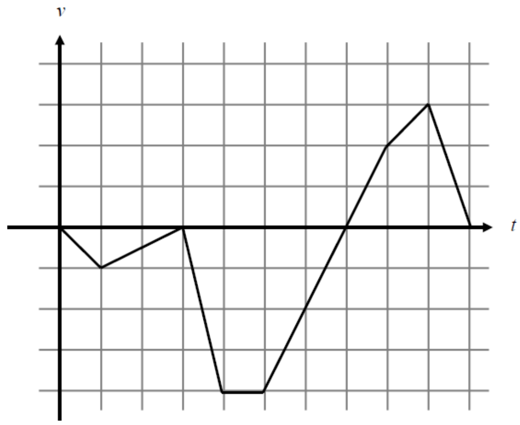
b) When will The Emperor hit the ground?

c) What is The Emperor's velocity at impact? Make sure to indicate the units in your answer.

d) What is The Emperor's speed at impact?

e) Find The Emperor's acceleration as a function of time. Make sure to indicate the units in your answer.

Example 4: Suppose the graph below shows the velocity of a particle moving along the x-axis. Justify each response to the following questions.



a) Which way does the particle move first?

b) When does the particle stop?

c) When does the particle change direction?

d) When is the particle moving left?

e) When is the particle moving right?

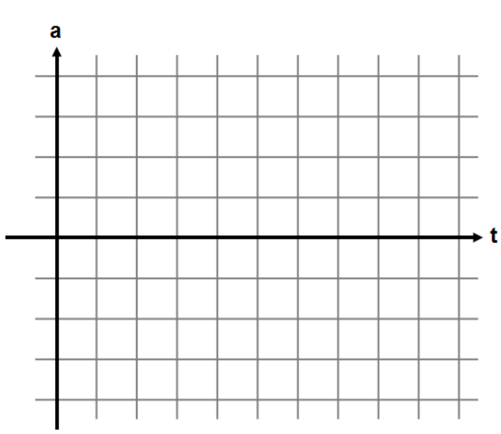
f) When is the particle speeding up?

g) When is the particle slowing down?

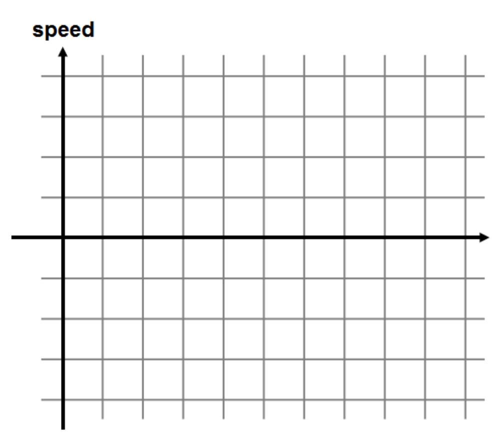
h) When is the particle moving the fastest?

i) When is the particle moving at a constant speed?

j) Graph the particle's acceleration over the interval $0 < t < 10$.



k) Graph the particle's speed over the interval $0 < t < 10$.



Determining When Speed is Increasing or Decreasing

The speed of an object is increasing when it is moving away from the x-axis of a velocity-time graph. The speed is decreasing when the object is moving towards the x-axis. If you do not have a graph, Speed is increasing when velocity and acceleration have the same sign. Speed is decreasing when velocity and acceleration have different signs.

Sign of the Velocity	Sign of the Acceleration	Speed
+	+	Increasing
-	-	Increasing
+	-	Decreasing
-	+	Decreasing

Derivatives in Economics

Economists use calculus to determine the rate of change of costs with respect to certain factors.The underlying principle behind the following definitions is that if x changes by 1 unit, then the change in y is approximately the value of the derivative at the original x.

Profit: Revenue – Cost or $P = R \cdot C$
Marginal Cost: The extra cost of producing one more item.
Marginal Revenue: The extra revenue from producing one more item.
Marginal Profit: The extra profit from producing one more item.

These marginal values can be found by finding the rate of change of the corresponding function at the current production amount.

Example 4: Suppose the daily cost, in dollars, of producing x Lego X-Wing kits is given by the function $C(x) = 0.002x^3 + 0.1x^2 + 42x + 300$, and currently there are 40 kits produced daily.

a) What is the current daily cost?

b) What would the actual additional daily cost of increasing production to 41 kits daily?

c) What is the marginal cost of the 41st kit?

Example 5: Suppose the cost of producing x units is given by the function $C(x) = 4x^2 + \frac{300}{x}$. What is the marginal cost of producing the 11th unit?

AB Calculus: Derivatives of Trigonometric Functions **Name: ______________________________**

Using the derivatives of $\sin x$ and $\cos x$, you can find the derivatives of the other 4 trig functions as well. Let's start by finding the derivative of the sine function using the limit definition of derivative.

Example 1: Find $f'(x)$ if $f(x) = \sin x$ using the limit definition of derivative.

a) First, let's look at some background information we will need:

You should already know this limit: $\lim_{x\to 0} \frac{\sin x}{x} =$ ______.

Investigate the following limit $\lim_{x\to 0} \frac{\cos x - 1}{x}$. What do you think the value of this limit is?

b) To prove $\lim_{x\to 0} \frac{\cos x - 1}{x}$ algebraically, multiply the top and bottom by $(\cos x + 1)$, then evaluate the limit.

c) Find the derivative of $\sin x$ using the limit definition of the derivative.

You can prove $\frac{d}{dx}[\cos x] = -\sin x$ using the same method and the same two limits above.

Here are the derivatives of all six basic trigonometric functions.

Derivatives of the Six Basic Trigonometric Functions

$\frac{d}{dx}[\sin x] = \cos x$	$\frac{d}{dx}[\cos x] = -\sin x$	$\frac{d}{dx}[\tan x] = \sec^2 x$
$\frac{d}{dx}[\cot x] = -\csc^2 x$	$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\frac{d}{dx}[\csc x] = -\csc x \cot x$

Using the derivatives of $\sin x$ and $\cos x$, the previous derivative rules we learned, and some algebraic massaging will enable us to prove the value of the other 4 derivatives.

Example 2: Prove the derivative of $\tan x$ is $\sec^2 x$.

Proving the derivative of $\cot x$ is $-\csc^2 x$ can be done in a similar way.

Example 3: Prove the derivative of $\sec x$ is $\sec x \tan x$.

Proving the derivative of $\csc x$ is $-\csc x \cot x$ can be done in a similar way.

Example 4: Find the derivative of each of the following functions

a) $f(x) = x^2 \sin x$

b) $f(x) = \frac{\cos x}{x}$

c) $g(t) = \sqrt{t} + \sec t$

d) $h(s) = \frac{1}{s} - 10 \csc s$

e) $y = x \cot x$

Using the Calculator to Graph the Derivative

You can use your calculator to graph the derivative for you using the procedures outlined below. This is not a required skill for success in this course, just something else your calculator can do. If you have a TI-84, this process will be much quicker.

We will be using the nDeriv(function, except we will be using it to define a function under Y_1. Remember the syntax is:

TI-83+ or TI-84 with Older Operating System	TI-84 Family with Newer Operating System
The **nDeriv(** function works as follows: **nDeriv(function, x, value)**	The **nDeriv(** function works as follows: $\frac{d}{d\mathbf{x}}(\mathbf{function})\Big\vert_{x=\mathbf{value}}$

For our example, let's use $\cos x$. The difference will be that our function will be entered as a function of x, and differentiated with respect to the same variable, and evaluated at the value of x instead of the actual number.

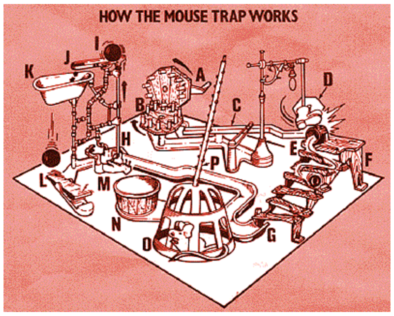
So we want to enter the following:

TI-83+ or TI-84 with Older Operating System	TI-84 Family with Newer Operating System
The **nDeriv(** function works as follows: $\boldsymbol{Y_1} =$ **nDeriv(** $\mathbf{cos}\ \boldsymbol{x}, \boldsymbol{x}, \boldsymbol{x}$**)**	The **nDeriv(** function works as follows: $\boldsymbol{Y_1} = \frac{\boldsymbol{d}}{\boldsymbol{d}\mathbf{x}}(\mathbf{cos\ x})\Big\vert_{x=\mathbf{x}}$

Example 5 Graph the derivative of $f(x) = \ln x$. What function does this look like? Graph your guess on the same screen.

Example 6 Graph the derivative of $f(x) = e^x$. What does this function look like? Graph your guess on the same screen.

AB Calculus: The Chain Rule **Name: ______________________________**



In the board game mousetrap, players are tasked with building a contraption that will capture the other mice in the game. To win, a player starts a chain reaction by turning a crank that moves a boot that kicks a marble that goes down a chute causing another ball to fall out of a tub and hit a lever that flips a man into a bucket, causing the trap to come down. The Chain Rule actually gets its name because it is a similar chain reaction whereby one action triggers another, which triggers another, which triggers another.

In the image above, the turning of the crank at point A travels all the way through the board, eventually trapping the mouse at point P. If we wanted to know the change of P with respect to A, We would need to start by finding the rate of change of B, the next stage in the chain after A, and work our way all the way through to P.

$$\frac{dP}{dA} = \frac{dB}{dA}\cdot\frac{dC}{dB}\cdot\frac{dD}{dC}\cdot\frac{dE}{dD}\cdot\frac{dF}{dE}\cdot\frac{dG}{dF}\cdot\frac{dH}{dG}\cdot\frac{dI}{dH}\cdot\frac{dJ}{dI}\cdot\frac{dK}{dJ}\cdot\frac{dL}{dK}\cdot\frac{dM}{dL}\cdot\frac{dN}{dM}\cdot\frac{dO}{dN}\cdot\frac{dP}{dO}$$

The Chain Rule is our weapon for deriving composite functions, or functions (other than just plain x) within other functions. Here are some examples of the types of functions the chain rule will allow us to differentiate.

Do Not Need Chain Rule	Need Chain Rule
$y = x^2 - 1$	$y = \sqrt{x^2 - 1}$
$y = \sin x$	$y = \sin 5x$
$y = 3x - 2$	$y = (x^2 - 1)^7$
$y = x - \tan x$	$y = x - \tan(x^2)$

The Chain Rule

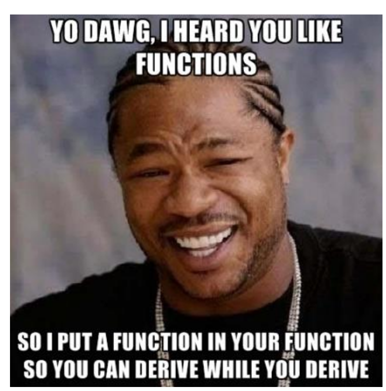
If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} \text{ OR } \frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

Composite functions have an "inside function" and an "outside function." Another way to look at this would be:

$$\frac{d}{dx}[f(g(x))] = f'(g(x))\,g'(x)$$

Derivative of the "outside function" ... leave the "inside function" alone. × Derivative of the "inside function"



The toughest part at first is identifying the "inside" and the "outside" functions.

Example 1: Identify the inside and outside function in each of the following composite functions.

a) $f(x) = \cos(5x^2)$

I $5x^2$

O $\cos x$

b) $g(x) = \sqrt{3x+1}$

I $3x+1$

O $\sqrt{x}$

c) $f(x) = \sin^2 x = (\sin x)^2$

I $\sin x$

O x^2

d) $f(x) = \dfrac{1}{5x+1}$

I $5x+1$

O $\dfrac{1}{x}$

Example 2: Find the derivative of each of the following functions.

a) $f(x) = \cos(5x^2)$

$\cos x \xrightarrow{d/dx} -\sin x$

$5x^2 \rightarrow 10x$

$f'(x) = -\sin(5x^2) \cdot 10x$

b) $g(x) = \sqrt{3x+1}$

$\sqrt{x} \xrightarrow{d/dx} \dfrac{1}{2\sqrt{x}}$

$3x+1 \rightarrow 3$

$g'(x) = \dfrac{1}{2\sqrt{3x+1}} \cdot 3$

c) $f(x) = \sin^2 x$ $(\sin x)^2$

$x^2 \xrightarrow{d/dx} 2x$

$\sin x \rightarrow \cos x$

$f'(x) = 2\sin x \cos x$

d) $f(x) = \dfrac{1}{5x+1}$

$\dfrac{1}{x} \xrightarrow{d/dx} -\dfrac{1}{x^2}$

$5x+1 \rightarrow 5$

$f'(x) = \dfrac{-1}{(5x+1)^2} \cdot 5$

Deriving and identifying the inside and outside functions gets tougher when you have functions with multiple inside functions. When this happens, you peel the function like an onion, taking the derivative of the outside of the remaining function that has not been derived yet and keeping the inside functions the same until you work your way all the way to the innermost function.

Example 3: Identify the functions that make up the composite functions below from inside to outside.

a) $f(x) = \sin^3 \sqrt{4x+1}$

$x^3 \xrightarrow{d/dx} 3x^2$

$\sin x \rightarrow \cos x$

$\sqrt{x} \rightarrow \dfrac{1}{2\sqrt{x}}$

$4x+1 \rightarrow 4$

$f'(x) = 3(\sin\sqrt{4x+1})^2 \cdot \cos\sqrt{4x+1} \cdot \dfrac{1}{2\sqrt{4x+1}} \cdot 4$

b) $g(x) = \dfrac{3}{\sqrt{\tan(4x^2+1)}}$

$\sqrt{x} \rightarrow x^{\frac{1}{2}} \xrightarrow{d/dx} \frac{1}{2}x^{-\frac{1}{2}}$ $3x^{-1} \xrightarrow{d/dx} -3x^{-2}$

$\dfrac{3}{x} \rightarrow \dfrac{-3}{x^2}$

$\sqrt{x} \rightarrow \dfrac{1}{2\sqrt{x}}$

$\tan x \rightarrow \sec^2 x$

$4x^2+1 \rightarrow 8x$

$g'(x) = \dfrac{-3}{(\sqrt{\tan(4x^2+1)})^2} \cdot \dfrac{1}{2\sqrt{\tan(4x^2+1)}} \cdot \sec^2(4x^2+1) \cdot 8x$

Example 4: Find the derivative of each of the following

a) $f(x) = \sin^3\sqrt{4x+1}$

b) $g(x) = \dfrac{3}{\sqrt{\tan(4x^2+1)}}$

Combining All the Rules

Example 5: Find the derivative of each of the following

$(2x-3)^{\frac{1}{2}} \to \frac{d}{dx} \to \frac{1}{2}(2x-3)^{-\frac{1}{2}}\,2$

a) $f(x) = (x^2+1)\sqrt{2x-3}$ $\quad \to (2x-3)^{\frac{1}{2}}$

$f'(x) = (2x)\cdot\sqrt{2x-3} + (x^2+1)\,\frac{1}{2}\,\frac{1}{\sqrt{2x-3}}\cdot 2$

b) $g(t) = \left(\dfrac{t-2}{2t+1}\right)^9$

$g'(t) = 9\left(\dfrac{t-2}{2t+1}\right)^8 \cdot \left(\dfrac{(2t+1)(1)-(t-2)(2)}{(2t+1)^2}\right)$

simplify inside: $\dfrac{2t+1-2t+4}{(2t+1)^2} \to \dfrac{5}{(2t+1)^2}$

Final: $\dfrac{9\cdot 5(t-2)^8}{(2t+1)^{10}}$ $\quad \to 45$

Example 6: For each of the following, use the fact that $g(5) = -3$, $g'(5) = 6$, $h(5) = 3$, and $h'(5) = -2$ to find $f'(5)$, if possible. If it is not possible, state what additional information is needed to find the value.

a) $f(x) = g(x)h(x)$

$f'(x) = g(x)\cdot h'(x) + g'(x)\,h(x) \to f'(5) = g(5)\,h'(5) + g'(5)\,h(5) = (-3)(-2) + (6)(3) = 6+18 = 24$

b) $f(x) = g\big(h(x)\big)$

$g(x) \to g'(x)$, $\quad h(x) \to h'(x)$

$f'(x) = g'(h(x))\,h'(x) \to f'(5) = g'(h(5))\,h'(5) = g'(3)\,h'(5) \Rightarrow$ Not possible, we need $g'(3)$

c) $f(x) = \dfrac{g(x)}{h(x)}$

$f'(x) = \dfrac{h(x)\,g'(x) - g(x)\,h'(x)}{(h(x))^2} \to f'(5) = \dfrac{h(5)\,g'(5) - g(5)\,h'(5)}{(h(5))^2} = \dfrac{(3)(6)-(-3)(-2)}{3^2} = \dfrac{18-6}{9} = \dfrac{12}{9} = \dfrac{4}{3}$

d) $f(x) = [g(x)]^3$

$x^3 \xrightarrow{d/dx} 3x^2$, $\quad g(x) \to g'(x)$

$f'(x) = 3(g(x))^2\,g'(x) \to f'(5) = 3(g(5))^2\,g'(5) = 3(-3)^2\,6 = 3\cdot 9\cdot 6 = 27\cdot 6 = 162$

Note: you may see the expression $[g(x)]^3$ as $g^3(x)$.