## AB Calculus: Velocity and Other Rates of Change  Name: ____________________

### Instantaneous Rates of Change

We have already seen that the instantaneous rate of change at a point is the same as the slope of the tangent line at a point a.k.a the derivative at that point. From now on, unless we use the phrase "average rate of change," we will assume that in calculus the phrase "rate of change" refers to the instantaneous rate of change.

If $f(x)$ represents a quantity, then $f'(x)$ represents the instantaneous rate of change of that quantity. $f(x)$ may describe a particle's position or it's velocity. In fact, $f(x)$ can represent any quantity such as the area of a circle, the temperature outside, the amount of rainfall in a region or the number of people infected with a disease. This is the true power of the derivative: it gives us a way to discuss how fast anything is changing at a single moment in time.

**Example 1:** An ice cream company knows that the cost, $C$ (in dollars), to produce $q$ quarts of cookie dough ice cream is a function of $q$, so $C = f(q)$. our "x" is q our "y" is C

a) If $f(200) = 70$, what are the units of the 200? What are the units of the 70? Clearly explain what the statement is telling you.

quarts $

it costs $70 to produce 200 qts of ice cream

b) If $f'(200) = 3$, what are the units of the 200? What are the units of the 3? Clearly explain what the statement is telling you. derivative of f HAS to have units of $\frac{\text{output}}{\text{input}}$

200 → quarts

3 → $/qt

when you're producing 200 quarts the cost is increasing at a rate of $3/quart

**Example 2:** The temperature $T$, in degrees Fahrenheit, of a cold pizza placed in a hot oven is given by $T = f(t)$, where $t$ is the time in minutes since the pizza was put in the oven.

a) What is the meaning of the statement $f(20) = 255$ in this scenario?

After being in oven for 20 minutes, the temperature of the pizza is 255°F

b) What is the sign of $f'(t)$? Why?

positive, why → T is increasing

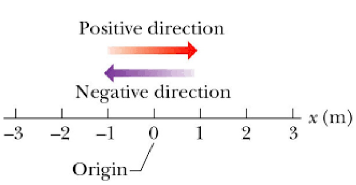
c) What are the units of $f'(20)$?

$\frac{\text{units of } T}{\text{units of } t} \rightarrow \frac{°F}{\text{min}}$

d) What is the meaning of the statement $f'(20) = 2$?

when pizza has been in oven for 20 minutes, The temperature is rising at a rate of 2°F/min

### Motion along a Line




There is perhaps no more important type of rate of change than that of motion. After all, it was Newton's motivation for inventing calculus in the first place. We have already looked at motion briefly when we investigated the basketball shot from a World Record height. To delve into this topic further, we will need to get familiar with some terminology and relationships between position, velocity, and acceleration.

**Relationships between Position, Velocity, and Acceleration**

Position functions, often denoted $s(t)$, give the position of an object at any point in time.

The **displacement** of an object is the **total change in position**. If we were given a position function $s(t)$ and an interval of time $[a, b]$, then the displacement would be $s(b) - s(a)$.

The **average velocity** of the object is the **total change in position (displacement) divided by the total change in time**. It can be thought of as the slope of the line connecting two points on a position function.

$$Average\ Velocity = \frac{displacement}{elapsed\ time} = \frac{s(b) - s(a)}{b - a}$$

The **instantaneous velocity** of an object is the derivative of the position function of the object.

$$Instantaneous\ Velocity = v(t) = s'(t) = the\ slope\ of\ the\ tangent\ line\ at\ any\ time\ t$$

Velocity is a quantity that gives both how fast something is moving as well as direction of the movement. If velocity is positive, then movement is occurring in the positive direction. If velocity is negative, then movement is occurring in the negative direction.

**Speed** is the **absolute value of the velocity**, therefore, speed is always positive.

$$Speed = |v(t)| = |s'(t)|$$

**Acceleration** is the rate of change of velocity or **the derivative of velocity**. Since it is the derivative of velocity, it is also **the second derivative of position**.

$$Acceleration = v'(t) = s''(t) = the\ slope\ of\ the\ tangent\ line\ to\ the\ velocity\ function\ at\ any\ time\ t$$

The **average acceleration** of an object is the **total change in velocity divided by the total change in time**. It can be thought of as the slope of the line connecting two points on a velocity function.

**Example 3:** To save his son Luke Skywalker from certain death, Darth Vader picks up The Emperor and throws him off of a ledge on the Death Star. As he falls, The Emperor's position $s$ is given by the function $s(t) = -16t^2 + 16t + 320$, where $s$ is measured in feet and $t$ is measured in seconds.

$s(2) = -16(2)^2 + 16(2) + 320 = -64 + 32 + 320$

a) What is The Emperor's displacement from $t = 1$ second to $t = 2$ seconds?

$s(2) - s(1) = -32\ ft$

$s(1) = -16(1)^2 + 16(1) + 320 = -16 + 16 + 320$

b) When will The Emperor hit the ground?

$0 = -16t^2 + 16t + 320$ / $0 = -16(t^2 - t - 20) \rightarrow 0 = -16(t+4)(t-5) \rightarrow t = -4$, $\boxed{t = 5\ sec}$

c) What is The Emperor's velocity at impact? Make sure to indicate the units in your answer.

Notation → $s'(5)$ or $v(5)$ / $s'(t) = -32t + 16$ / $s'(5) = -32(5) + 16 = \boxed{-144\ ft/s}$

d) What is The Emperor's speed at impact?

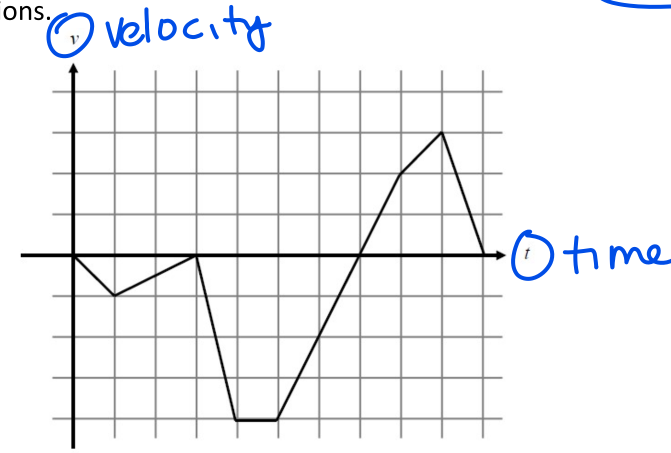
$144\ ft/s$ (speed $= |v(t)|$)

e) Find The Emperor's acceleration as a function of time. Make sure to indicate the units in your answer.

$a(t) = v'(t) = s''(t)$ / $a(t) = -32\ ft/s/s$ $(ft/s^2)$

moving Horizontally

**Example 4:** Suppose the graph below shows the velocity of a particle moving along the x-axis. Justify each response to the following questions.




a) Which way does the particle move first?

Left $v(t) < 0$

b) When does the particle stop?

$t = 3, 7, 10$ $v(t) = 0$

c) When does the particle change direction?

$t = 7$ $v(t)$ changes sign

d) When is the particle moving left?

$(0,3)$ $(3,7)$

e) When is the particle moving right?

$(7,10)$

f) When is the particle speeding up?

$(0,1)$ $(3,4)$ $(7,9)$

g) When is the particle slowing down?

$(1,3)$ $(5,7)$ $(9,10)$

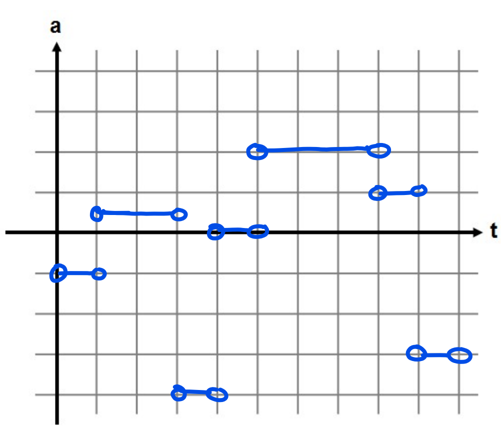
h) When is the particle moving the fastest?

$(4,5)$

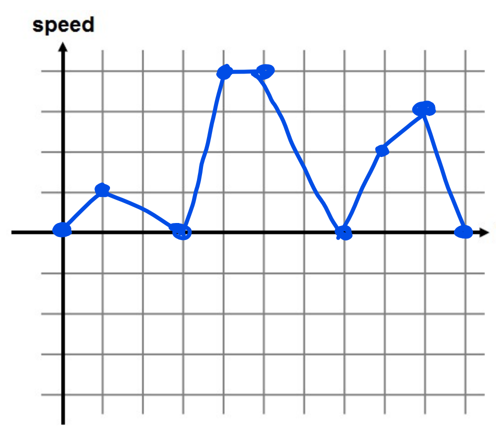
i) When is the particle moving at a constant speed?

$(4,5)$

j) Graph the particle's acceleration over the interval $0 < t < 10$.




k) Graph the particle's speed over the interval $0 < t < 10$.



**Determining When Speed is Increasing or Decreasing**

The speed of an object is increasing when it is moving away from the x-axis of a velocity-time graph. The speed is decreasing when the object is moving towards the x-axis. If you do not have a graph, Speed is increasing when velocity and acceleration have the same sign. Speed is decreasing when velocity and acceleration have different signs.

| Sign of the Velocity | Sign of the Acceleration | Speed |
|---|---|---|
| + | + | Increasing |
| - | - | Increasing |
| + | - | Decreasing |
| - | + | Decreasing |

**Derivatives in Economics**

Economists use calculus to determine the rate of change of costs with respect to certain factors.The underlying principle behind the following definitions is that if $x$ changes by 1 unit, then the change in y is approximately the value of the derivative at the original $x$.

**Profit:** Revenue – Cost or $P = R \cdot C$
**Marginal Cost:** The extra cost of producing one more item.
**Marginal Revenue:** The extra revenue from producing one more item.
**Marginal Profit:** The extra profit from producing one more item.

These marginal values can be found by finding the rate of change of the corresponding function at the current production amount.

**Example 4:** Suppose the daily cost, in dollars, of producing $x$ Lego X-Wing kits is given by the function $C(x) = 0.002x^3 + 0.1x^2 + 42x + 300$, and currently there are 40 kits produced daily.

a) What is the current daily cost?

b) What would the actual additional daily cost of increasing production to 41 kits daily?

c) What is the marginal cost of the $41^{st}$ kit?

**Example 5:** Suppose the cost of producing $x$ units is given by the function $C(x) = 4x^2 + \frac{300}{x}$. What is the marginal cost of producing the $11^{th}$ unit?

**AB Calculus: Derivatives of Trigonometric Functions** **Name:** ______________________________

Using the derivatives of $\sin x$ and $\cos x$, you can find the derivatives of the other 4 trig functions as well. Let's start by finding the derivative of the sine function using the limit definition of derivative.

**Example 1:** Find $f'(x)$ if $f(x) = \sin x$ using the limit definition of derivative.

a) First, let's look at some background information we will need:

You should already know this limit: $\lim_{x\to 0} \frac{\sin x}{x} = \underline{1}$.

Investigate the following limit $\lim_{x\to 0} \frac{\cos x - 1}{x}$. What do you think the value of this limit is? $\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$

b) To prove $\lim_{x\to 0} \frac{\cos x - 1}{x}$ algebraically, multiply the top and bottom by $(\cos x + 1)$, then evaluate the limit.

$$\frac{\cos x - 1}{x} \cdot \frac{\cos x + 1}{\cos x + 1} = \frac{\cos^2 x - 1}{x(\cos x + 1)} = \frac{-\sin^2 x}{x(\cos x + 1)} = \frac{\sin x}{x} \cdot \frac{-\sin x}{\cos x + 1}$$

$$\lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{-\sin x}{\cos x + 1} = 1 \cdot \frac{-\sin 0}{\cos 0 + 1} = 1 \cdot \frac{0}{1+1} = 0$$

c) Find the derivative of $\sin x$ using the limit definition of the derivative.

$f(x) = \sin x$ what $f'(x)$?

$$\lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$\lim_{h\to 0} \frac{\sin x \cdot \cos h + \sin h \cos x - \sin x}{h}$$

$$\lim_{h\to 0} \frac{\sin x \cos h - \sin x}{h} + \frac{\sin h \cos x}{h}$$

$$\lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \frac{\sin h \cos x}{h}$$

$$0 + 1 \cdot \cos x$$

$$\boxed{f'(x) = \cos x}$$

You can prove $\frac{d}{dx}[\cos x] = -\sin x$ using the same method and the same two limits above.

Here are the derivatives of all six basic trigonometric functions.

**Derivatives of the Six Basic Trigonometric Functions**

| | | |
|---|---|---|
| $\frac{d}{dx}[\sin x] = \cos x$ | $\frac{d}{dx}[\cos x] = -\sin x$ | $\frac{d}{dx}[\tan x] = \sec^2 x$ |
| $\frac{d}{dx}[\cot x] = -\csc^2 x$ | $\frac{d}{dx}[\sec x] = \sec x \tan x$ | $\frac{d}{dx}[\csc x] = -\csc x \cot x$ |

Using the derivatives of $\sin x$ and $\cos x$, the previous derivative rules we learned, and some algebraic massaging will enable us to prove the value of the other 4 derivatives.

**Example 2:** Prove the derivative of $\tan x$ is $\sec^2 x$.

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2}$$

$$= \frac{\cos^2 x + \sin^2 x}{(\cos x)^2} = \frac{1}{(\cos x)^2} = \boxed{\sec^2 x}$$

Proving the derivative of $\cot x$ is $-\csc^2 x$ can be done in a similar way.

**Example 3:** Prove the derivative of $\sec x$ is $\sec x \tan x$.

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{(\cos x)(0) - (1)(-\sin x)}{(\cos x)^2}$$

$$= \frac{0 + \sin x}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}\,\frac{\sin x}{\cos x} = \boxed{\sec x \tan x}$$

Proving the derivative of $\csc x$ is $-\csc x \cot x$ can be done in a similar way.

**Example 4:** Find the derivative of each of the following functions

a) $f(x) = x^2 \sin x$

$$f'(x) = 2x \sin x + x^2 \cos x$$

b) $f(x) = \dfrac{\cos x}{x}$

$$f'(x) = \frac{x(-\sin x) - \cos x(1)}{x^2} = \frac{-x \sin x - \cos x}{x^2}$$

c) $g(t) = \sqrt{t} + \sec t$

$$g(t) = t^{\frac{1}{2}} + \sec t \;/\; g'(t) = \tfrac{1}{2} t^{\frac{-1}{2}} + \sec t \tan t$$

d) $h(s) = \dfrac{1}{s} - 10 \csc s$

$$h(s) = 1s^{-1} - 10 \csc s \;/\; h'(s) = -1s^{-2} + 10 \csc s \cot s$$

e) $y = x \cot x$

$$\frac{dy}{dx} = x \cdot (-\csc^2 x) + 1 \cot x$$

$$= -x \csc^2 x + \cot x$$