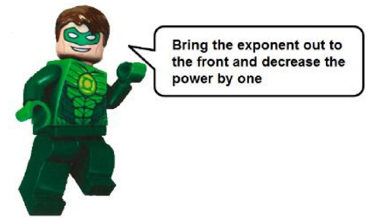
AB Calculus: Rules for Differentiation **Name:** ______________________________

Power Rule for Differentiation



If n is any number, then $\frac{d}{dx}[x^n] = n \cdot x^{n-1}$, provided x^{n-1} exists.

The key to using the power rule is to get comfortable using exponent rules to write a function as a power of x.

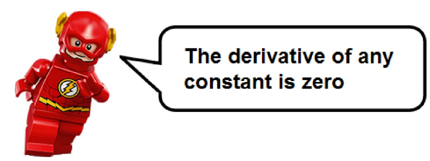
Example 1: Find the derivative of each of the following.

a) $f(x) = x^5$ $f'(x) = 5x^4$

b) $f(x) = \sqrt[3]{x^2}$ $f(x) = x^{2/3}$ $f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$ or $\frac{2}{3\sqrt[3]{x}}$

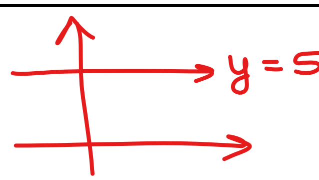
c) $f(x) = \frac{1}{x^4}$ $f(x) = 1x^{-4}$ $f'(x) = -4x^{-5}$ or $\frac{-4}{x^5}$

The Derivative of a Constant Function



If c is any constant value, then $\frac{d}{dx}[c] = 0$.

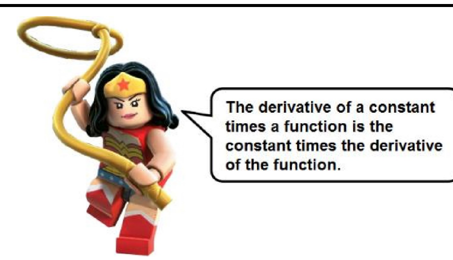
Example 2: Let $f(x) = 5$. Find $f'(x)$.



(slope of a horizontal line is 0)

$f'(x) = 0$

The Constant Multiple Rule for Derivatives



If u is a differentiable function of x and c is a constant value, then

$$\frac{d}{dx}[cu] = c\frac{du}{dx}$$

Example 3: Find the derivative of each of the following.

a) $f(x) = 5x^7$ $\longrightarrow$ $5 \cdot x^7$ so $\frac{d}{dx}x^7 = 7x^6$

$f'(x) = 35x^6$ and so $5 \cdot 7x^6 = 35x^6$

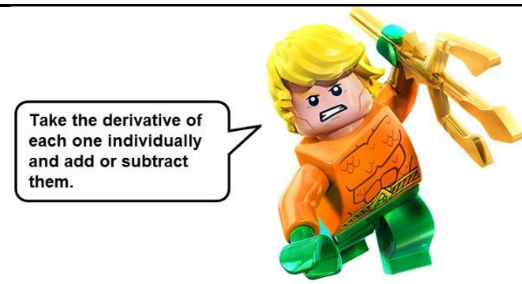
b) $f(x) = \frac{4}{5x^3}$

$f(x) = \frac{4}{5}x^{-3}$ $f'(x) = -\frac{12}{5}x^{-4}$ OR $\frac{-12}{5x^4}$

The Sum and Difference Rule for Derivatives

If u and v are differentiable functions of x, then wherever u and v are differentiable

$$\frac{d}{dx}[u \pm v] = \frac{du}{dx} \pm \frac{dv}{dx}$$



Example 3: Find the derivative of each of the following.

a) $f(x) = x^3 + 4x^2 - 2x + 7$

$f'(x) = 3x^2 + 8x - 2$

b) $f(x) = \frac{3}{(-2x)^4} - \frac{x}{2} + \frac{1}{4}$

$f(x) = \frac{3}{(-2)^4 x^4} - \frac{1}{2}x + \frac{1}{4}$

$\to f(x) = \frac{3}{16}x^{-4} - \frac{1}{2}x + \frac{1}{4}$

$\boxed{f'(x) = \frac{-12}{16}x^{-5} - \frac{1}{2}}$

Example 4: Find the equation of the tangent line to the function $f(x) = 4x^3 - 6x + 5$ when $x = 2$.

$f'(x) = 12x^2 - 6$

$f'(2) = 12(2)^2 - 6$

$= 12\,4 - 6$

$= 48 - 6 = 42$

$\boxed{y - 25 = 42(x-2)}$

$f(2) = 4(2)^3 - 6(2) + 5$

$= 4(8) - 12 + 5$

$= 32 - 12 + 5$

$= 20 + 5 = 25$

Example 5: Let $h(x) = (4x^2 + 1)(2x - 5)$. Find $h'(x)$.

$h(x) = 8x^3 - 20x^2 + 2x - 5$

$\boxed{h'(x) = 24x^2 - 40x + 2}$

Example 6: The volume of a cube with sides of length s is given by the formula $V = s^3$. Find $\frac{dV}{ds}$ when $s = 4$cm.

$\frac{dV}{ds} \to$ the derivative of volume with respect to length

$\frac{dV}{ds} = 3s^2$

$\left.\frac{dV}{ds}\right|_{s=4} = 3(4)^2 = \boxed{48\text{ cm}^2}$

We have already seen that the derivative of the sum of two functions is the sum of the derivatives of the two functions. This does not work for the product or quotient of two functions. To illustrate this, look at the following example:

Example 7: $\frac{d}{dx}[x^2 \cdot 3x]$

$x^2\, 3x = 3x^3$

$\frac{d}{dx}(3x^3) = 9x^2$

$f(x) = x^2 \quad g(x) = 3x$

$f'(x) = 2x \quad g'(x) = 3$

$x^2 \cdot 3 + 2x \cdot 3x = 3x^2 + 6x^2$

$= 9x^2$

* Never * ever Mistake

$\frac{d}{dx}(f(x) \cdot g(x))$

$= f'(x)\, g'(x)$

$2x \cdot 3$

$6x$

Product Rule

$\frac{d}{dx}[f(x) \cdot g(x)] = f(x)\, g'(x) + f'(x) \cdot g(x)$

The Product Rule for Derivatives

If u and v are differentiable functions of x, then

$$\frac{d}{dx}[u \cdot v] = u \cdot \frac{dv}{dx} + v\frac{du}{dx}$$

Example 8: Find $\frac{dy}{dx}$ of each of the following functions.

a) $y = (3 + 2\sqrt{x})(5x^3 - 7)$

① distribute

$y = 15x^3 - 21 + 10x^{3.5} - 14x^{\frac{1}{2}}$

$\frac{dy}{dx} = 45x^2 + 35x^{2.5} - 7x^{-\frac{1}{2}}$

② $y = (3 + 2x^{\frac{1}{2}})(5x^3 - 7)$

$\frac{dy}{dx} = (3 + 2x^{\frac{1}{2}})(15x^2) + (x^{-\frac{1}{2}})(5x^3 - 7)$

$= 45x^2 + 30x^{2.5} + 5x^{2.5} - 7x^{-\frac{1}{2}}$

b) $y = (3x - 2x^2)(4 + 5x)$

$y = 12x + 15x^2 - 8x^2 - 10x^3$

$\frac{dy}{dx} = 12 + 30x - 16x - 30x^2$

$\frac{dy}{dx} = (3 - 4x)(4 + 5x) + (5)(3x - 2x^2)$

$= 12 + 15x - 16x - 20x^2 + 15x - 10x^2$

Note: It is also valid to multiply out the function first and then take the derivative.

The Quotient Rule for Derivatives

If u and v are differentiable functions of x, then

$\frac{\text{Lo dHi} - \text{Hi dLo}}{\text{Lo}^2}$

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v \cdot \frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example 9: Find the derivative of each of the following.

a) $f(x) = \frac{x}{x^2 + 1}$

$f'(x) = \frac{(x^2+1)(1) - (x)(2x)}{(x^2+1)^2} = \frac{x^2 + 1 - 2x^2}{(x^2+1)^2} = \frac{1 - x^2}{(x^2+1)^2}$

b) $f(x) = \frac{5x^2}{x^3 + 1}$

$f'(x) = \frac{(x^3+1)(10x) - (5x^2)(3x^2)}{(x^3+1)^2}$

$= \frac{10x^4 + 10x - 15x^4}{(x^3+1)^2} = \boxed{\frac{10x - 5x^4}{(x^3+1)^2}}$

Second and Higher Order Derivatives

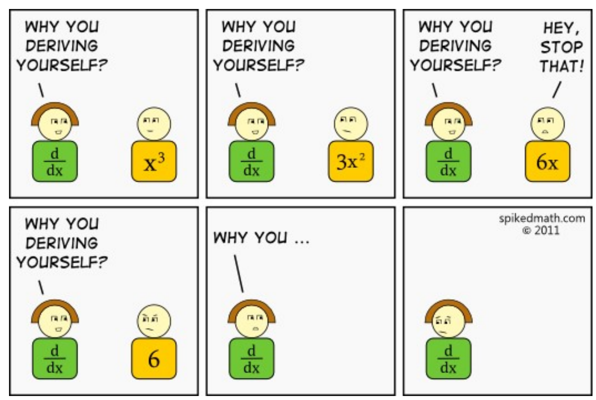
The first derivative of y with respect to x is denoted y', $f'(x)$, or $\frac{dy}{dx}$. The second derivative with respect to x is denoted y'', $f''(x)$, or $\frac{d^2y}{dx^2}$. The second derivative is an example of a higher order derivative. We can continue to take derivatives (as long as they exist) using the following notation:

$$\frac{d}{dx}\frac{dy}{dx} = \frac{d^2y}{dx^2}$$

First Derivative	y'	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}[f(x)]$
Second Derivative	y''	$f''(x)$	$\frac{d^2y}{dx^2}$	$\frac{d^2}{dx^2}[f(x)]$
Third Derivative	y'''	$f'''(x)$	$\frac{d^3y}{dx^3}$	$\frac{d^3}{dx^3}[f(x)]$
Fourth Derivative	$y^{(4)}$	$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$	$\frac{d^4}{dx^4}[f(x)]$
Nth derivative	$y^{(n)}$	$f^{(n)}(x)$	$\frac{d^ny}{dx^n}$	$\frac{d^n}{dx^n}[f(x)]$

Example 10: Find the indicated derivative of each of the following.

a) $\frac{d^4}{dx^4}[-5x^8 + 2x^6 - 9x^3 + 32x - 1]$



Find the 4th derivative

$\frac{d}{dx} \rightarrow -40x^7 + 12x^5 - 27x^2 + 32$

$\frac{d^2}{dx^2} \rightarrow -280x^6 + 60x^4 - 54x$

$\frac{d^3}{dx^3} \rightarrow -1680x^5 + 240x^3 - 54$

$\boxed{\frac{d^4}{dx^4} \rightarrow -8400x^4 + 720x^2}$

b) $\frac{d^2}{dx^2}\left[\frac{x}{x-1}\right]$

$\frac{d}{dx} \rightarrow \frac{(x-1)(1)-(x)(1)}{(x-1)^2} \rightarrow \frac{x-1-x}{(x-1)^2} \rightarrow \frac{-1}{(x-1)^2}$

$\frac{d^2}{dx^2} \rightarrow \frac{(x-1)^2(0)-(-1)\ 2(x-1)^1\cdot 1}{(x-1)^4} \rightarrow \frac{2(x-1)}{(x-1)^4} \rightarrow \frac{2}{(x-1)^3}$

c) Find $\frac{d^5y}{dx^5}$ if $\frac{d^4y}{dx^4} = 2\sqrt{x}$

$\frac{d}{dx}\left[2x^{\frac{1}{2}}\right] = \boxed{1x^{-\frac{1}{2}} \text{ or } \frac{1}{\sqrt{x}}}$