AB Calculus Exponential Growth Notesheet **Name: ____**

For exponential growth functions, the more you have, the more you get. For exponential decay functions, the less you have, the less you lose. Quantities that grow/decay by a factor or percentage at regular intervals are exponential. This can be stated equivalently as:

The rate of change of a quantity is directly proportional to the quantity itself.
Mathematically, we state this as $\frac{dy}{dt} = ky$, where k is either a growth or decay constant.

Example 1 Solve the separable differential equation $\frac{dy}{dt} = ky$

$$\int \frac{dy}{y} = \int k\,dt \rightarrow e^{\ln|y|} = e^{kt+c}$$

$$\boxed{y = ce^{kt}}$$

dy/dt

Example 2 Bacteria in a culture increased from 400 to 1600 in three hours. Assuming that the rate of increase of the population is directly proportional to the population, y

a) Find an appropriate equation to model the population.

$$1600 = 400e^{(k)(3)}$$

$$4 = e^{3k} \rightarrow \ln 4 = 3k$$

$$k = \tfrac{1}{3}\ln 4$$

$$\boxed{y = 400e^{(\frac{1}{3}\ln 4)t}}$$

b) Find the number of bacteria at the end of six hours. (calculator)

$$y(6) = 400e^{(\frac{1}{3}\ln 4)6} = 6{,}400$$

$\frac{dy}{dt}$ y

Example 3 Radium-226 loses its mass at a rate that is directly proportional to its mass. If its half-life is 1590 years, and if we start with a sample of radium-226 with a mass of 100 mg,

a) Find the formula for the mass that remains after t years.

$$y = ce^{kt}$$

$$50 = 100e^{k(1590)} \rightarrow \frac{1}{2} = e^{k(1590)}$$

$$\ln\tfrac{1}{2} = 1590k \rightarrow \frac{\ln(\frac{1}{2})}{1590}$$

$$\boxed{y = 100e^{\frac{\ln(\frac{1}{2})}{1590}t}}$$

b) Find the mass after 1000 years. (calculator)

$$y(1000) = 64\,666 \text{ mg}$$

c) When will the mass be reduced to 30 mg? (calculator)

$$30 = 100e^{\frac{\ln(\frac{1}{2})}{1590}t}$$

$$\boxed{t = 2{,}761775 \text{ yrs}}$$

$$\frac{30}{100} = e^{\frac{\ln(\frac{1}{2})}{1590}t} \quad \ln(.3) = \frac{\ln(\frac{1}{2})}{1590}t \quad t = \frac{1590\ln(.3)}{\ln(\frac{1}{2})}$$

If by hand | or just graph

Example 4 Suppose the amount of oil pumped from one of the canyon wells in Whittier, California decreases at a continuous rate of 10% per year. When will the well's output fall to one-fifth of its present level? (calculator)

$\frac{dy}{dt} = -.1t$

$k = -.1$

16.094 yrs

$y = ce^{-.1t}$

$10 = 50e^{-.1t} \rightarrow \frac{1}{5} = e^{-.1t} \rightarrow \frac{\ln\left(\frac{1}{5}\right)}{-.1} = \frac{-.1t}{-.1} \rightarrow t =$

Example 5 Suppose that a population of fruit flies grows in proportion to the number of fruit flies in the population. If there were 100 fruit flies after the second day and 500 flies after the fourth day, how many flies were in the original population? (calculator)

$\frac{dy}{dt} = ky$

$y = ce^{kt}$

$100 = ce^{k(2)}$

$500 = ce^{k(4)}$

$c = \frac{100}{e^{2k}}$

$c = \frac{500}{e^{4k}}$

$\frac{100}{e^{2k}} = \frac{500}{e^{4k}}$

$100e^{4k} = 500e^{2k}$

$e^{4k} = 5e^{2k}$

$\frac{e^{4k}}{e^{2k}} = 5$

$e^{4k-2k} = 5$

$e^{2k} = 5$

$2k = \ln 5 \rightarrow \boxed{k = \frac{\ln 5}{2}}$

Part 2

$100 = ce^{(\frac{1}{2}\ln 5)(2)}$

$100 = ce^{\ln 5}$

$100 = 5c \rightarrow$ so $\boxed{c = 20}$

Newton's Law of Cooling

$$\boxed{\frac{dT}{dt} = k(T - T_s)}$$

COOL

Newton's Law of Cooling states that the rate of change in the temperature of an object is proportional to the difference between the object's temperature and the temperature in the surrounding medium.

Example 6 Find the general solution to the differential equation for Newton's Law of Cooling.

$\frac{dT}{dt} = k(T - T_s)$

$\int \frac{dT}{T - T_s} = \int k\,dt$

$e^{\ln|T - T_s|} = e^{kt + c}$

$T - T_s = ce^{kt} \rightarrow \boxed{T = ce^{kt} + T_s}$

Example 7 As part of his summer job at a restaurant, Jim learned to cook up a big pot of soup late at night, just before closing time, so that there would be plenty of soup to feed customers the next day. He also found out that, while refrigeration was essential to preserve the soup overnight, the soup was too hot to be put directly into the fridge when it was ready. (The soup had just boiled at 100˚C, and the fridge was not powerful enough to accommodate a big pot of soup if it was any warmer than 20 ˚C). Jim discovered that by cooling the pot in a sink full of cold water, (kept running, so that its temperature was roughly constant at 5 ˚C) and stirring occasionally, he could bring the temperature of the soup to 60 ˚C in ten minutes. How long before closing time should the soup be ready so that Jim could put it in the fridge and leave on time? (calculator)

$T(t) = 20$

$T_s = 5$

$T(0) = 100$

$T(10) = 60$

$T = ce^{kt} + T_s$

$100 = ce^{k(0)} + 5$

$c = 95$

$T = 95e^{kt} + 5$

$60 = 95e^{k(10)} + 5$

$55 = 95e^{10k}$

$\frac{55}{95} = e^{10k}$

$\ln\left(\frac{55}{95}\right) = 10k$

$k = \frac{1}{10}\ln\left(\frac{55}{95}\right)$

Final equation

$T = 95e^{\frac{1}{10}\ln\left(\frac{55}{95}\right)t} + 5$

$20 = 95e^{\frac{1}{10}\ln\left(\frac{55}{95}\right)t} + 5$

when $t = 33.773$ min