AB Calculus Fundamental Theorem of Calculus Day 2 Notesheet **Name:** ______________________

$\sin x \to \cos x$

$x^3 \to 3x^2$

$\ln x \to \frac{1}{x}$

$\sqrt{x} \to \frac{1}{2\sqrt{x}}$

To Begin: Answer all parts of the following three questions.

1. Find the derivative of the following

   a) $f(x) = 4\sin(x^3)$

   $f'(x) = 4\cos(x^3)\, 3x^2$

   b) $f(x) = \ln(\sqrt{4x+2})$

   $4x+2 \to 4$

   $f'(x) = \frac{1}{\sqrt{4x+2}} \cdot \frac{1}{2\sqrt{4x+2}} \cdot 4$

2. Evaluate the following Integrals.

   a) $\int_2^3 (x^4 + 4x + 6)dx$

   $\left[\frac{1}{5}x^5 + \frac{4}{2}x^2 + 6x\right]_2^3$

   $\left(\frac{1}{5}(3)^5 + 2(3)^2 + 6(3)\right) - \left(\frac{1}{5}(2)^5 + 2(2)^2 + 6(2)\right)$

   b) $\int_\pi^{2\pi} \left(\frac{1}{x} - \sin x\right) dx$

   $\left[\ln|x| + \cos x\right]_\pi^{2\pi}$

   $= (\ln(2\pi) + \cos(2\pi)) - (\ln(\pi) + \cos(\pi))$

   $= \ln(2\pi) + 1 - \ln\pi - (-1)$

   $= \ln\left(\frac{2\pi}{\pi}\right) + 2 = \boxed{2 + \ln 2}$

   c) $\int \left(\frac{3}{\sqrt[3]{x^2}} + \frac{5}{1+x^2}\right) dx$

   $3\int x^{-\frac{2}{3}} dx + 5\int \frac{1}{1+x^2} dx$

   $= 3\,3x^{\frac{1}{3}} + 5\tan^{-1} x + C$

   d) $\int (7^x + \csc^2 x + 2)dx$

   $\frac{1}{\ln 7} \cdot 7^x - \cot x + 2x + C$

3. Given that $f(x) = \frac{1}{3}x^3 + x^2$, find the following:

   a) Find the x-coordinates of the critical points of f(x). Justify your answer.

   $f'(x) = 0$ or $f'(x)$ undefined

   $f'(x) = x^2 + 2x$

   $0 = x^2 + 2x \to 0 = x(x+2)$

   $x = 0,\ x = -2$

   b) Find the x-coordinate of any relative minima or maxima. Justify your answer.

   f': $+$ at -2 $-$ at 0 $+$

   rel max at $x = -2$ since f' changes $+$ to $-$

   rel min at $x = 0$ since f' changes $-$ to $+$

   c) Find the open intervals where the function is increasing and decreasing. Justify each response.

   Inc $(-\infty, -2)$ $(0, \infty)$ $f' > 0$

   Dec $(-2, 0)$ $f' < 0$

   d) Find any points of inflection. Justify your answer.

   Given concavity changes

   f'': $-$ at -1 $+$

   $f''(x) = 2x + 2$

   $0 = 2x + 2 \to x = -1$

   P.O.I at $x = -1$ since f'' changes $-$ to $+$

   e) Find the open intervals where the function is concave up and concave down. Justify each response.

   Ccup $(-1, \infty)$ $f'' > 0$

   Ccdown $(-\infty, -1)$ $f'' < 0$

Using the evaluation part of the Fundamental Theorem of Calculus we learned last time, we are going to develop the concept of the other part of the Fundamental Theorem of Calculus. We are going to determine how to take a derivative of a function that is defined as an integral and discuss what it means to define a function as an integral. Once we can do both of these things, we can answer all the same types of questions about increasing, decreasing, concave up, concave down, and inflection points that we did earlier in the year.

Ex 3: Find $\int_2^x g'(t)\,dt$

$g(t)\Big|_2^x$

$= g(x) - g(2)$

Ex 4: Find $\boxed{\frac{d}{dx}\left[\int_2^x g'(t)\,dt\right]}$ → what we did on ex 3

$\frac{d}{dx}\left(g(x) - g(2)\right)$

$g'(x)$

The Second Fundamental Theorem of Calculus
If f is continuous on an open interval containing a, then, for every x in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

$\frac{d}{dx}\int_{-2}^x f(t)\,dt$

$\frac{d}{dx} F(t)\Big|_{-2}^x$

$\frac{d}{dx}\left(F(x) - F(-2)\right)$

Ex 5: If $g(x) = \int_{-2}^x f(t)\,dt$, then $g'(x) =$

$f(x)$

Ex 6: Find $\frac{d}{dx}\left[\int_3^x (5t^3 - 4t + 1)\,dt\right]$

$5x^3 - 4x + 1$

Ex 7: $\frac{d}{dx}\left[\int_{g(x)}^{f(x)} h'(t)\,dt\right]$ $\int_{g(x)}^{f(x)} h'(t)\,dt = h(t)\Big|_{g(x)}^{f(x)} = h(f(x)) - h(g(x))$

$$\frac{d}{dx}\left[h(f(x)) - h(g(x))\right] = h'(f(x))\,f'(x) - h'(g(x))\,g'(x)$$

The Second Fundamental Theorem of Calculus Extended
If f is continuous on an open interval containing a, then, for every x in the interval,

$$\frac{d}{dx}\left[\int_{h(x)}^{g(x)} f(t)\,dt\right] = f\big(g(x)\big)g'(x) - f\big(h(x)\big)h'(x)$$

Ex 8: Find $\frac{d}{dx}\left[\int_{x^2}^{2x} f(t)\,dt\right]$

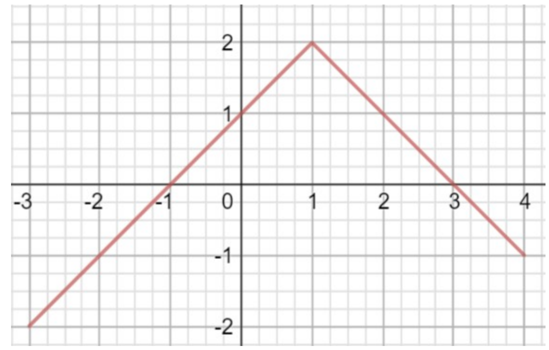
$= f(2x)\cdot 2 - f(x^2)\cdot 2x$

Ex 9: Let $g(x) = \int_{6x}^{4x^2} \sqrt{1+t^4}\,dt$. Find $g'(x)$.

$g'(x) = \sqrt{1+(4x^2)^4} \cdot 8x - \sqrt{1+(6x)^4}\;6$

Now that we have established how to take a derivative of an integral function, let's look more closely at what it means to define a function as an integral. $g'(x) = w(x)$

Ex 10: Let $g(x) = \int_{-2}^{x} w(t)\, dt$ where the graph of $w(t)$ is given below.



a) Find $g(0)$.

$g(0) = \int_{-2}^{0} w(t)\, dt = 0$

b) Find $g(2)$.

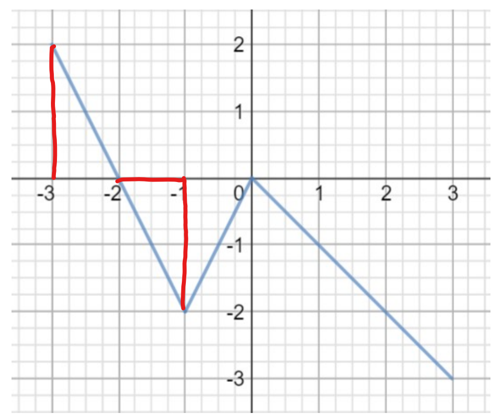
$g(2) = \int_{-2}^{2} w(t)\, dt = 3$

c) Find $g(-3)$.

$g(-3) = \int_{-2}^{-3} w(t)\, dt = 1.5$

Now put it all together …

Ex 11: Suppose the function below is the graph of $f(t)$ and $\boxed{g(x) = \int_{-1}^{x} f(t)\, dt.}$ $\rightarrow$ $g'(x) = f(x)$



a) Complete the table.

x	-3	-2	-1	0	1	2	3
g(x)	0	1	0	-1	-1.5	-3	-5.5

$g(-3) = \int_{-1}^{-3} f(t)\, dt$ $\quad g(-2) = \int_{-1}^{-2}$

b) What are the intervals on which g is increasing or decreasing? Justify each response.

g inc $(-3, -2)$ $g' > 0$

g dec $(-2, 0)\ (0, 3)$ or $(-2, 3)$ $g' < 0$

c) What are the intervals on which g is concave up or concave down? Justify each response.

$g''(x) = f'(x)$ g is ccup $(-1, 0)$ $f' > 0$

g is ccdown $(-3, -1)\ (0, 3)$ $f' < 0$

d) For what value of x does g have a relative minimum or maximum? Justify your response.

g has no rel minimum $\quad$ g' never changes − to +

g has rel. max at $x = -2$ $\quad$ $f(x) = 0$ and changes + to −

e) For what value of x does g have an inflection point? Justify your response.

POI at $x = -1$ and $x = 0$ since g'' changes signs

f) Graph g(x).