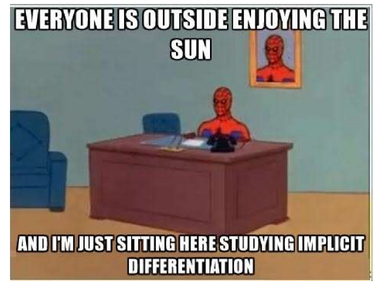
AB Calculus: Implicit Differentiation **Name:** ______________________________



Equations that are solved for y are called **explicit** functions, whereas equations that are not solved for y are called **implicit** functions. For instance, the equation $x + 2y - 3 = 0$ implies that y is a function of x, even though it is not written in the form $y = -\frac{1}{2}x + \frac{3}{2}$. Up to this point in this class we have been using explicit functions of y expressed in the form $y = f(x)$ such as

Find $\frac{dy}{dx}$ here is easy, because y is already isolated

$$\frac{d}{dx}\left[y = \frac{x+1}{x+2}\right] \quad \text{or} \quad y = \sin x$$

$\rightarrow \frac{dy}{dx} =$ Do the quotient rule

If we have an equation that involves both x and y in which y has not been solved for x, then we say the equation defines y as an implicit function of x. In this case, we may (or may not) be able to solve for y in terms of x to obtain an explicit function (or possibly several functions).

Example 1 Find the derivative of the following expressions

always 1 so not important

a) Find $\frac{d}{dx}$ of x^3 $\rightarrow 3x^2 \cdot \cancel{\frac{dx}{dx}}$

b) Find $\frac{d}{dx}$ of y^3 $\rightarrow 3y^2 \frac{dy}{dx}$

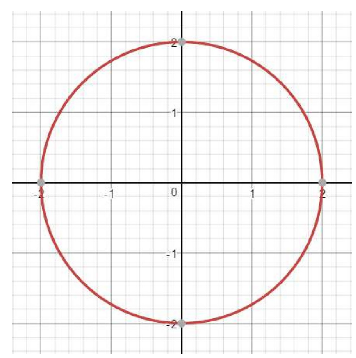
Find $\frac{d}{dx}$ of xy

$1y + x\frac{dy}{dx}$

Find $\frac{d}{dx}$ of $x^3 + y^2 - 3xy$

$3x^2 + 2y\frac{dy}{dx} - 3\left(1y + x\frac{dy}{dx}\right)$

Example 2 The graph to the right is a circle with the equation $x^2 + y^2 = 4$.



a) Solve the equation for y and find the derivative of the resulting function(s)

$y^2 = 4 - x^2$

$\frac{d}{dx}\left[y = \pm\sqrt{4-x^2}\right]$

$\frac{dy}{dx} = \pm\frac{1}{2}(4-x^2)^{-\frac{1}{2}}(-2x)$

$\boxed{\frac{dy}{dx} = \frac{\pm x}{\sqrt{4-x^2}}}$

b) Find the derivative by differentiating implicitly

Implicit → don't solve for y 1st

$\frac{d}{dx}\left[x^2 + y^2 = 4\right] \rightarrow 2x + 2y\frac{dy}{dx} = 0$ solve for dy/dx

$\frac{2y\frac{dy}{dx}}{2y} = \frac{-2x}{2y}$

$\boxed{\frac{dy}{dx} = \frac{-x}{y}}$

c) Find where the derivative is positive, where it is negative, where it is zero, and where it is undefined.

Q2, Q4

Q1, Q3

$x=0$

$y=0$

If we have y written as an explicit function of x, $y = f(x)$, then we can find the derivative using the rules we have previously learned. For an equation which defines y as an implicit function of x, we can compute the derivative without solving for y in terms of x with the following procedure. The key to this entire procedure is to remember that even though you did not (or cannot) write y as a function of x, y is implicitly defined as a function of x.

Steps for Implicit Differentiation

1. Differentiate both sides of the equation with respect to x (y is a function of x, so use chain rule).
2. Collect all $\frac{dy}{dx}$ terms on the left side of the equation and move all other terms to the right side.
3. Factor $\frac{dy}{dx}$ out of the left side of the equation if there is more than one $\frac{dy}{dx}$ term.
4. Solve for $\frac{dy}{dx}$ (It is okay to have both x's and y's in your answer).

Note: To find $\frac{dy}{dx}$ at a given point, take the derivative of both sides then immediately plug in the point and simplify.

Example 4 Given the curve $x^3 + y^3 = 6xy$,

a) Find $\frac{dy}{dx}$

$$\frac{d}{dx}\left[x^3+y^3=6xy\right]$$

$$3x^2+3y^2\frac{dy}{dx}=6\left(1y+x\frac{dy}{dx}\right)$$

$$3x^2+3y^2\frac{dy}{dx}=6y+6x\frac{dy}{dx}$$

$$-3x^2-6x\frac{dy}{dx} = -3x^2-6x\frac{dy}{dx}$$

$$3y^2\frac{dy}{dx}-6x\frac{dy}{dx}=6y-3x^2$$

$$\frac{dy}{dx}\left(3y^2-6x\right)=6y-3x^2$$

$$\boxed{\frac{dy}{dx}=\frac{6y-3x^2}{3y^2-6x}}$$

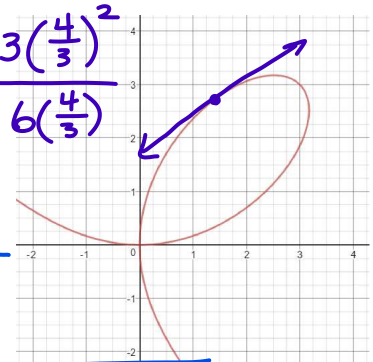
b) Find the equation of the tangent line and normal line to the graph at the point $\left(\frac{4}{3},\frac{8}{3}\right)$.

b.) Tangent Line

$$\frac{dy}{dx}\Big|_{\left(\frac43,\frac83\right)}=\frac{6\left(\frac83\right)-3\left(\frac43\right)^2}{3\left(\frac83\right)^2-6\left(\frac43\right)}$$

$$=\frac{\frac{48}{3}-\frac{16}{3}}{\frac{64}{3}-\frac{24}{3}}$$

$$=\frac{32}{40}=\frac45$$



TL

$$\boxed{y-\frac83=\frac45\left(x-\frac43\right)}$$

Normal

$$y-\frac83=-\frac54\left(x-\frac43\right)$$

Example 5 Find $\frac{dy}{dx}$ at the point $(0, 0)$ of the function $\tan(x + y) = x$.

$$\sec^2(x+y)\cdot\left(1+\frac{dy}{dx}\right)=1$$

$$1+\frac{dy}{dx}=\frac{1}{\sec^2(x+y)}$$

$$\boxed{\frac{dy}{dx}=\frac{1}{\sec^2(x+y)}-1}$$

$$\frac{dy}{dx}\Big|_{(0,0)}=\frac{1}{\sec^2(0+0)}-1$$

$$\boxed{\frac{dy}{dx}\Big|_{0,0}=0}$$

Example 6 Find $\frac{d^2y}{dx^2}$ of $2x^3 - 3y^2 = 8$

$$6x^2-6y\frac{dy}{dx}=0$$

$$\frac{6x^2}{6y}=\frac{6y\frac{dy}{dx}}{6y}$$

$$\boxed{\frac{dy}{dx}=\frac{x^2}{y}}$$

Now, $\frac{d}{dx}\left[\frac{dy}{dx}=\frac{x^2}{y}\right]$

$$\frac{d^2y}{dx^2}=\frac{y\,2x-x^2\frac{dy}{dx}}{y^2}$$

$$\frac{d^2y}{dx^2}=\frac{\frac{y}{y}2xy-x^2\left(\frac{x^2}{y}\right)}{y^2}$$

$$\boxed{\frac{d^2y}{dx^2}=\frac{2xy^2-x^4}{y^3}}$$