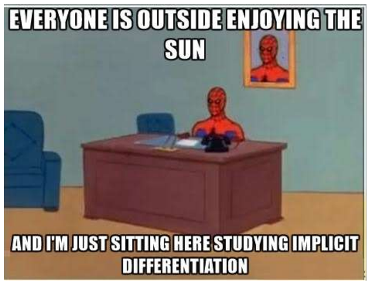
AB Calculus: Implicit Differentiation **Name:** ______________________

Equations that are solved for y are called **explicit** functions, whereas equations that are not solved for y are called **implicit** functions. For instance, the equation $x + 2y - 3 = 0$ implies that y is a function of x, even though it is not written in the form $y = -\frac{1}{2}x + \frac{3}{2}$. Up to this point in this class we have been using explicit functions of y expressed in the form $y = f(x)$ such as



$$y = \frac{x+1}{x+2} \qquad \text{or} \qquad y = \sin x$$

If we have an equation that involves both x and y in which y has not been solved for x, then we say the equation defines y as an implicit function of x. In this case, we may (or may not) be able to solve for y in terms of x to obtain an explicit function (or possibly several functions).

Example 1 Find the derivative of the following expressions

a) Find $\frac{d}{dx}$ of x^3 $\rightarrow 3x^2$

b) Find $\frac{d}{dx}$ of y^3 $\rightarrow 3y^2 \cdot \frac{dy}{dx}$

c) Find $\frac{d}{dx}$ of xy

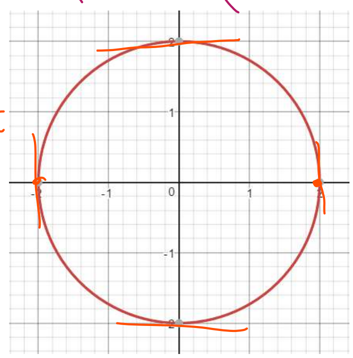
with respect to "x"

$1 \cdot y + x \cdot \frac{dy}{dx}$

d) Find $\frac{d}{dx}$ of $x^3 + y^2 - 3(xy)$

$= 3x^2 + 2y\frac{dy}{dx} - 3\left(1 \cdot y + x \cdot \frac{dy}{dx}\right)$

Example 2 The graph to the right is a circle with the equation $x^2 + y^2 = 4$.



a) Solve the equation for y and find the derivative of the resulting function(s)

$x^2 + y^2 = 4$

$y^2 = 4 - x^2$

$y = \pm\sqrt{4-x^2}$

$\frac{d}{dx} y = \frac{d}{dx} \pm\sqrt{4-x^2} \rightarrow (4-x^2)^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}(4-x^2)^{-\frac{1}{2}} \cdot (-2x)$

$\frac{dy}{dx} = \frac{-x}{\sqrt{4-x^2}} \rightarrow$ if $x = \pm 2 \rightarrow$ den $= 0 \rightarrow$ V Tangent

if $x = 0 \rightarrow$ num $= 0 \rightarrow$ H Tangent

b) Find the derivative by differentiating implicitly

$\frac{d}{dx}\left(x^2 + y^2 = 4\right)$

$2x + 2y \cdot \frac{dy}{dx} = 0$

$2y\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y} = \frac{dy}{dx}$

H.T. $\rightarrow -x = 0$

$x = 0$

V.T. $\rightarrow y = 0$

c) Find where the derivative is positive, where it is negative, where it is zero, and where it is undefined.

$\frac{dy}{dx} = 0$ if $x = 0$ (on the y-axis) / $\frac{dy}{dx}$ is undefined if $y = 0$ (the axis) / $\frac{dy}{dx} > 0$ if x & y are opp signs (Q2 & Q4)

$\frac{dy}{dx} < 0$ if x & y are same sign (Q1 & Q3)

If we have y written as an explicit function of x, $y = f(x)$, then we can find the derivative using the rules we have previously learned. For an equation which defines y as an implicit function of x, we can compute the derivative without solving for y in terms of x with the following procedure. The key to this entire procedure is to remember that even though you did not (or cannot) write y as a function of x, y is implicitly defined as a function of x.

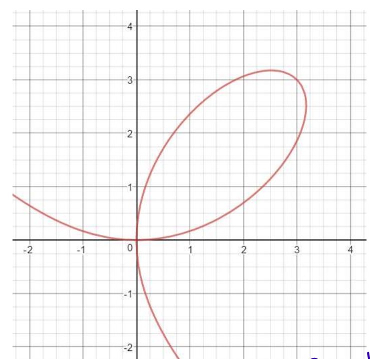
Steps for Implicit Differentiation

1. Differentiate both sides of the equation with respect to x (y is a function of x, so use chain rule).
2. Collect all $\frac{dy}{dx}$ terms on the left side of the equation and move all other terms to the right side.
3. Factor $\frac{dy}{dx}$ out of the left side of the equation if there is more than one $\frac{dy}{dx}$ term.
4. Solve for $\frac{dy}{dx}$ (It is okay to have both x's and y's in your answer).

Note: To find $\frac{dy}{dx}$ at a given point, take the derivative of both sides then immediately plug in the point and simplify.

take $\frac{d}{dx}$

Example 4 Given the curve $x^3 + y^3 = 6xy$,



a) Find $\frac{dy}{dx}$

$3x^2 + 3y^2\frac{dy}{dx} = 6\left(1y + x\frac{dy}{dx}\right)$

$3x^2 + 3y^2\frac{dy}{dx} = 6y + 6x\frac{dy}{dx}$

$-3x^2 \quad -6x\frac{dy}{dx} \qquad -6x\frac{dy}{dx} \quad -3x^2$

$3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$

$\frac{dy}{dx}(3y^2 - 6x) = 6y - 3x^2$, divide by $3y^2 - 6x$

$\frac{dy}{dx} = \frac{6y-3x^2}{3y^2-6x}$ or $\boxed{\frac{2y - x^2}{y^2 - 2x}}$

b) Find the equation of the tangent line and normal line to the graph at the point $\left(\frac{4}{3}, \frac{8}{3}\right)$.

Point-Slope $\left(x, y, \frac{dy}{dx}\right)$

$\left.\frac{dy}{dx}\right|_{\left(\frac{4}{3},\frac{8}{3}\right)} = \frac{2\left(\frac{8}{3}\right) - \left(\frac{4}{3}\right)^2}{\left(\frac{8}{3}\right)^2 - 2\left(\frac{4}{3}\right)} = \frac{\frac{3}{3}\cdot\frac{16}{3} - \frac{16}{9}}{\frac{64}{9} - \frac{8}{3}\cdot\frac{3}{3}} = \frac{48-16}{64-24} = \frac{32}{40} = \frac{4}{5}$

T $y - \frac{8}{3} = \frac{4}{5}\left(x - \frac{4}{3}\right)$

N $y - \frac{8}{3} = -\frac{5}{4}\left(x - \frac{4}{3}\right)$

Example 5 Find $\frac{dy}{dx}$ at the point $(0, 0)$ of the function $\tan(x + y) = x$.

$\sec^2(x+y)\cdot\left(1 + \frac{dy}{dx}\right) = 1$

$1 + \frac{dy}{dx} = \frac{1}{\sec^2(x+y)}$ so $\frac{dy}{dx} = \frac{1}{\sec^2(x+y)} - 1$ or $\boxed{\frac{dy}{dx} = \cos^2(x+y) - 1}$

$\left.\frac{dy}{dx}\right|_{(0,0)} = \cos^2(0+0) - 1 = \cos^2 0 - 1 = 1^2 - 1 = 1 - 1 = \boxed{0}$

Example 6 Find $\frac{d^2y}{dx^2}$ of $2x^3 - 3y^2 = 8$

1) $\frac{dy}{dx}$

$\Rightarrow 6x^2 - 6y\cdot\frac{dy}{dx} = 0$

$-6y\frac{dy}{dx} = -6x^2$

$\frac{dy}{dx} = \frac{-6x^2}{-6y}$ so $\frac{dy}{dx} = \frac{x^2}{y}$

2) $\frac{d^2y}{dx^2} = \frac{y\,2x - x^2\frac{dy}{dx}}{y^2} = \frac{\frac{y\,2xy - x^2 x^2}{y}}{y^2}$

$\boxed{= \frac{2xy^2 - x^4}{y^3}}$