AB Calculus: L'Hôpital's Rule **Name:** ______________________________

We are back to evaluating limits. Recall the indeterminate form we discussed earlier, $\frac{0}{0}$. There is another indeterminate form, $\pm\frac{\infty}{\infty}$. When direct substitution yields either of these indeterminate forms, we have had to employ various algebraic methods to compute the limit, because, even though the function value failed to exist there, the limit still could, and it could be anything. In this section, we will learn another way to evaluate limits that yield indeterminate form using the derivative.

Example 1 (Calculator) Investigate the following limits by using a table or graph and approximate the limit.

a) $\lim_{x\to 0}\left(\frac{e^{3x}-1}{x}\right) \rightarrow 3$

b) $\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right) \rightarrow 1/2$

c) $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$ ~ 7.38 or 50 .

Before we get too far into it, let's review some algebraic techniques for dealing with indeterminate forms.

Example 2 Evaluate the following limits.

a) $\boxed{\lim_{x\to -1}\frac{2x^2-2}{x+1}} \rightarrow \frac{0}{0}$ IND

$2(x-1) = 2x-2$

$\frac{2(x^2-1)}{x+1} \rightarrow \frac{2\cancel{(x+1)}(x-1)}{\cancel{x+1}}$ Plug in $x=-1$ again $\rightarrow 2(-1-1) = -4$

b) $\lim_{x\to\infty}\frac{3x^2-1}{2x^2+1} \rightarrow \frac{\infty}{\infty}$ IND

take the coefficients since at $x\to\infty$, we use EBM so $\lim_{x\to\infty} f(x) = \frac{3}{2}$

c) $\lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7} \rightarrow \frac{0}{0}$ IND

$\frac{(\sqrt{x+2}-3)(\sqrt{x+2}+3)}{(x-7)(\sqrt{x+2}+3)} \rightarrow \frac{\cancel{x-7}\ \ \cancel{x+2-9}}{\cancel{(x-7)}(\sqrt{x+2}+3)} \rightarrow \frac{1}{\sqrt{x+2}+3} \rightarrow \frac{1}{\sqrt{9}+3} = \frac{1}{3+3} = 1/6$

Not all indeterminate forms can be reconciled via algebraic techniques. Another method, discovered by Swiss mathematician Johann Bernoulli, is called L'Hôpital's Rule, named after the 17th century French mathematicial Guillaume de L'Hôpital who did not discover it, but published it in the first-ever textbook on differential calculus. This method uses the derivative to evaluate limits that yield indeterminate form.

L'Hôpital's Rule

If $\lim_{x\to a}\frac{f(x)}{g(x)}$ yields either of the indeterminate forms $\boxed{\frac{0}{0} \text{ or } \pm\frac{\infty}{\infty}}$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$.

do not confuse this with the quotient rule

Note: To use L'Hôpital's Rule, you must show that the limit yields either $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$, first!

Example 3 Evaluate each of the following limits.

a) $\lim_{x\to 0}\frac{e^{3x}-1}{x}$

$\lim_{x\to 0} e^{3x}-1=0 \rightarrow$ L'Hospital's $\lim_{x\to 0}\frac{e^{3x}-1}{x} = \lim_{x\to 0}\frac{e^{3x}(3)}{1}$

$\lim_{x\to 0} x = 0$

$= \frac{e^{3(0)}\,3}{1} = 3$

b) $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}$

$\lim_{x\to 0}\sqrt{1+x}-1=0$ L'Hospital $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} = \lim_{x\to 0}\frac{\frac{1}{2}(1+x)^{-1/2}}{1}$

$\lim_{x\to 0} x = 0$

$\boxed{=1/2}$

c) $\lim_{x\to 0}\frac{2\tan x}{5x}$

$\lim_{x\to 0} 2\tan x = 0$ L'Hospital's $\lim_{x\to 0}\frac{2\tan x}{5x} = \lim_{x\to 0}\frac{2\sec^2 x}{5} = 2/5$

$\lim_{x\to 0} 5x = 0$

d) $\lim_{x\to\infty}\frac{\ln x}{2\sqrt{x}}$

$\lim_{x\to\infty}\ln x \rightarrow \infty$ L'Hospital $\lim_{x\to\infty}\frac{\ln x}{2\sqrt{x}} = \lim_{x\to\infty}\frac{\frac{1}{x}}{\cancel{2}\cdot\frac{1}{\cancel{2}}x^{-1/2}}$

$\lim_{x\to\infty} 2\sqrt{x}\rightarrow\infty$

$\rightarrow \lim_{x\to\infty}\frac{1}{x}\cdot x^{1/2} \rightarrow \boxed{0}$

Sometimes we may need to repeat ourselves. Sometimes we may need to repeat ourselves.

Example 4 Evaluate each of the following limits.

a) $\lim_{x\to -\infty}\frac{x^2}{e^{-x}}$

$\lim_{x\to -\infty} x^2 = \infty$ L'Hospitals $\lim_{x\to -\infty}\frac{x^2}{e^{-x}} = \lim_{x\to -\infty}\frac{2x}{-e^{-x}} \rightarrow \frac{-\infty}{-\infty}$ L'H

$\lim_{x\to -\infty} e^{-x} = \infty$

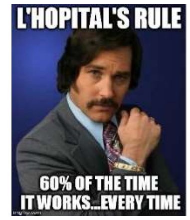
$-\frac{x}{2} = -\frac{1}{2}x$

$= \lim_{x\to -\infty}\frac{2}{e^{-x}} \rightarrow \boxed{0}$

b) $\lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2} \rightarrow \frac{0}{0} \rightarrow$ L'H $\rightarrow \frac{\frac{1}{2}(1+x)^{-\frac{1}{2}}-\frac{1}{2}}{2x} \rightarrow \frac{0}{0}$

$\rightarrow \frac{-\frac{1}{4}(1+x)^{-3/2}}{2} \rightarrow -1/8$

The rule works great, but it only works with the indeterminate forms $\frac{0}{0}$ and $\pm\frac{\infty}{\infty}$. There are other indeterminate forms including $0^0, 1^\infty, \infty-\infty, 0\cdot\infty$, and ∞^0. We can still use the rule, but we have to first convert them to $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$.



Example 6 Evaluate each of the following limits.

a) $\lim_{x\to\infty} e^{-x}\sqrt{x}$ $\to 0\cdot\infty$ is a weird indeterminate

$\lim_{x\to\infty} \frac{\sqrt{x}}{e^x} \to \frac{\infty}{\infty} \to \boxed{0}$

$\hookrightarrow$ L'H $\to \lim_{x\to\infty} \frac{\frac{1}{2\sqrt{x}}}{e^x} \to \frac{1}{2\sqrt{x}\cdot e^x} \to 0$

b) $\lim_{x\to1^+}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$ $\to \frac{x-1-\ln x}{(x-1)\ln x} \to \frac{0}{0} \to$ L'H $\to \frac{1-\frac{1}{x}}{(x-1)\frac{1}{x}+1\ln x}$

$\left(\frac{1}{0}-\frac{1}{0}\right)$

$(\infty-\infty)$

$-x^{-1}$

$1-\frac{1}{x}+\ln x$

$\to$ L'H $\frac{x^{-2}}{x^{-2}+\frac{1}{x}} \to \frac{1}{1+1} \to 1/2$

Sometimes we need logs to come to our rescue.

Example 7 Evaluate each of the following limits.

x is in the exponent, use ln

a) $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$ $\to 1^\infty$ IND

1) take ln of the problem

$x\ln\left(1+\frac{2}{x}\right) \to \frac{\ln\left(1+\frac{2}{x}\right)}{1/x} \to \frac{0}{0} \to$ L'H $\to \frac{\frac{(-2)1}{1+\frac{2}{x}}\cancel{\left(\frac{-2}{x^2}\right)}}{\cancel{\frac{-1}{x^2}}} \to +2$

2.) Fix step 1 e^2

b) $\lim_{x\to0^+} x^x$ $0^0 \to ?1$

1) take ln $\lim_{x\to0^+} x\ln x$

$\to \lim_{x\to0^+} \frac{\ln x}{1/x} \to \frac{-\infty}{\infty} \to$ L'H $\to \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \frac{x^2}{-x} = -x = 0$

Last step is $e^0 = \boxed{1}$

c) $\lim_{x\to\infty} x^{\frac{1}{x}}$ ∞^0 IND

1) "ln it" $\frac{1}{x}\ln x$

$\lim_{x\to\infty} \frac{\ln x}{x} \to \frac{\infty}{\infty} \to 0$ by inspection

So last step undo $e^0 = \boxed{1}$

EBM