AB Calculus: Intro to Limits **Name:** ______________________________

The limit is fundamental to the study of calculus. It is important to acquire a good working knowledge of the limit before moving forward, because you will find out through the duration of this course that really, it is all about limits.

Example 1: Use your calculator to generate a graph of $f(x) = \frac{x^2-4}{x-2}$; $x \neq 2$.

a) Why is $x \neq 2$ included in the function definition?

b) Complete the table of values below to determine what happens as x gets "close" to 2.

	x approaches 2 from the **left** →					←	x approaches 2 from the **right**			
x	1.5	1.75	1.9	1.99	1.999	2	2.001	2.01	2.1	2.25	2.5
$f(x)$											

Definition of a Limit

If the y-value of a function $f(x)$ becomes close to a single value L as x gets closer and closer to a point *c* from both the left and the right side, then the **limit of $f(x)$ as x approaches *c* is L**, which can be written using mathematical notation as:

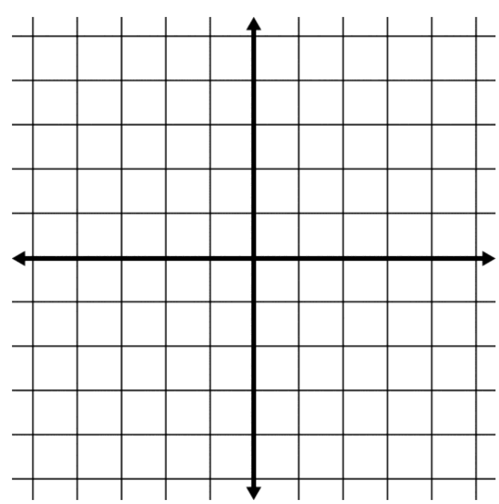
$$\lim_{x \to c} f(x) = L$$

c) Apply this definition to the function from above to find $\lim_{x \to 2} f(x)$.

The previous definition of a limit is an informal definition. The formal mathematical definition is called the epsilon-delta definition. It is not required for AP Calculus, however, if you would like to learn about it, click on, or type in, the following link or scan the QR code to the right. https://goo.gl/gyzBaK

Example 2: Use the graph to find $\lim_{x \to 2} g(x)$, where g is:

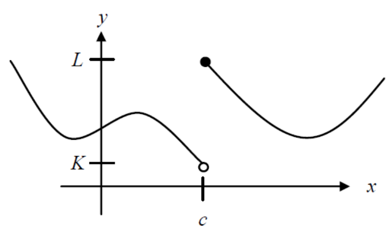
$$g(x) = \begin{cases} 1, & x \neq 2 \\ 0, & x = 2 \end{cases}$$



The last example can be confusing at first, but think of a damaged bridge crossing a canyon. Even though the bridge is broken and it seems like the missing section has fallen into the water below, you can probably tell where the missing section should be. The height of that point would be the value of the limit. Limits are important because they give us the ability to discuss what is going on at a point mathematically whether the bridge, or graph, exists at that point or not. With limits, you are only concerned with the y-value the graph approaches, not what the y-value actually is at that point.

One-Sided Limits

Suppose we have the graph of $f(x)$ below. Notice that the function below does not approach the same y-value as x approaches c from the left and right sides. When tracing the graph starting to the left of c, the graph approaches the y-value K. When tracing the graph starting to the right of c, the graph approaches the y-value L.



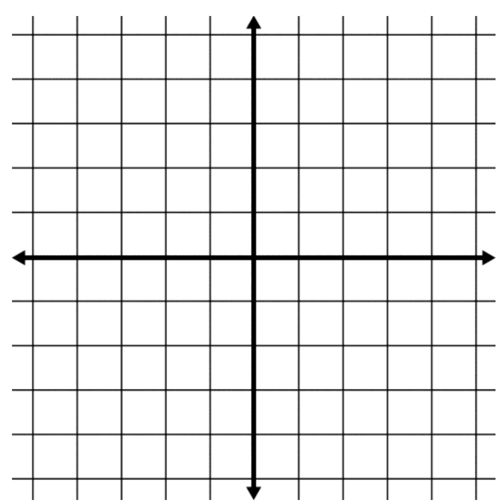
Sometimes we are only interested in what the function approaches as x approaches from the right or left of c. We can say this using the following notation:

$\lim_{x \to c^+} f(x) = L$ … "the limit of $f(x)$ as x approaches c from the right is L."
$\lim_{x \to c^-} f(x) = K$ … "the limit of $f(x)$ as x approaches c from the left is K."

The limit of a function as x approaches any number c exists if and only if the limit as x approaches c from the right is equal to the limit as x approaches c from the left. Using limit notation, we have:

$$\lim_{x \to c} f(x) \text{ exists} \Longleftrightarrow \lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x)$$

Example 3: Let $f(x) = \begin{cases} 5 - 2x, & x > 2 \\ x - 3, & x \le 2 \end{cases}$



a) Graph $f(x)$.

b) Find $\lim_{x \to 2^-} f(x)$.

c) Find $\lim_{x \to 2^+} f(x)$.

d) Find $\lim_{x \to 2} f(x)$?

When Limits Do Not Exist

There are times when a limit fails to exist, meaning that we are unable to find a value for the limit at the given point. When this happens, we say that the limit **does not exist**, abbreviating this statement as **DNE**. There are three types of common function behavior that lead to a limit that does not exist.

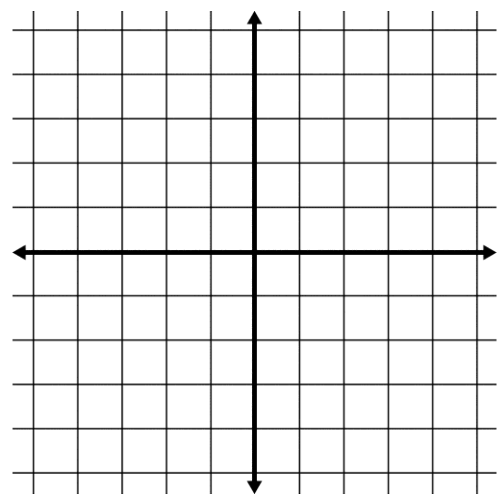
Common Types of Behavior Associated with a Limit that Does Not Exist

1. $f(x)$ approaches a different number from the right side of c than it approaches from the left side.
2. $f(x)$ increases or decreases without bound as x approaches c.
3. $f(x)$ oscillates between two fixed values as x approaches c.

Note: We will be more specific about Cause Number 2 when we explore infinite limits.

Example 4: Investigate the existence of the following limit using a graph and table.

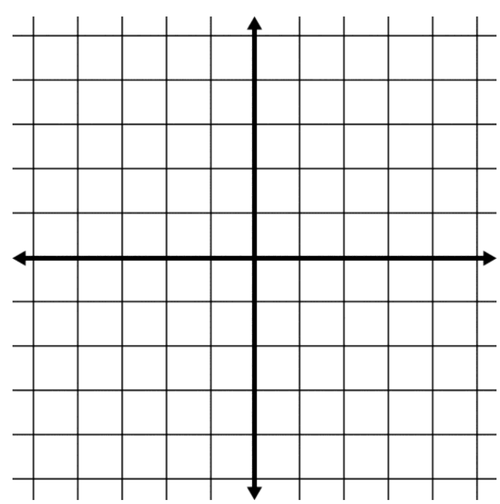
$$\lim_{x \to 0} \frac{|x|}{x}$$



x	-0.5	-0.25	-0.1	-.01	-.001	0	.001	.01	.1	.25	.5
$f(x)$											

Example 5: Investigate the existence of the following limit using a graph and table.

$$\lim_{x \to 1} \frac{1}{(x-1)^2}$$



x	0.5	0.75	0.9	0.99	0.999	1	1.001	1.01	1.1	1.25	1.5
$f(x)$											

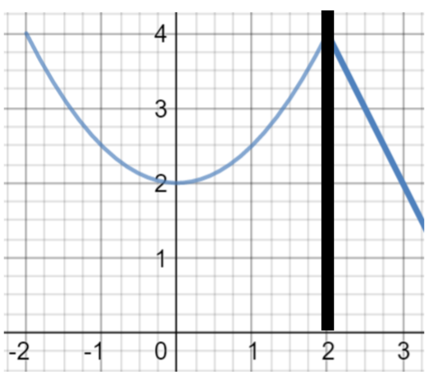
Example 6: Use a table to investigate $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$

x	$\frac{2}{\pi}$	$\frac{2}{3\pi}$	$\frac{2}{5\pi}$	$\frac{2}{7\pi}$	$\frac{2}{9\pi}$	$\frac{2}{11\pi}$	$\frac{2}{13\pi}$	As $x \to 0$
$f(x)$								

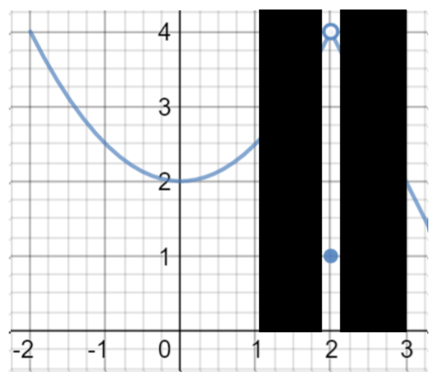
Finding Limits from Graphs

Hopefully you are getting comfortable with this new idea of a limit value and how it is different from a function value. When trying to find limits and function values from graphs, refer to the following visual aids:

When looking for a limit value at $x = c$, imagine that you have got a thick vertical line covering up $x = c$ with only the graph showing on either side of $x = c$. You are now looking to see what y-value the graph is approaching on either side of $x = c$. If the graphs appear to be approaching the same y-value, the limit exists and is that y-value. Otherwise, the limit does not exist.



When looking for a function value at $x = c$, imagine that you have got shudders on either side of $x = c$ with only a vertical sliver at $x = c$ visible between them. You are now looking for the dot or the piece of graph that exists in that narrow sliver. If it exists, the y-value of the dot is the function value $f(c)$. Otherwise, the function value is undefined.



Example 7: Use the graph to evaluate the following limits

A) $\lim_{x \to 0} f(x)$

B) $\lim_{x \to 6^-} f(x)$

C) $\lim_{x \to 2^-} f(x)$

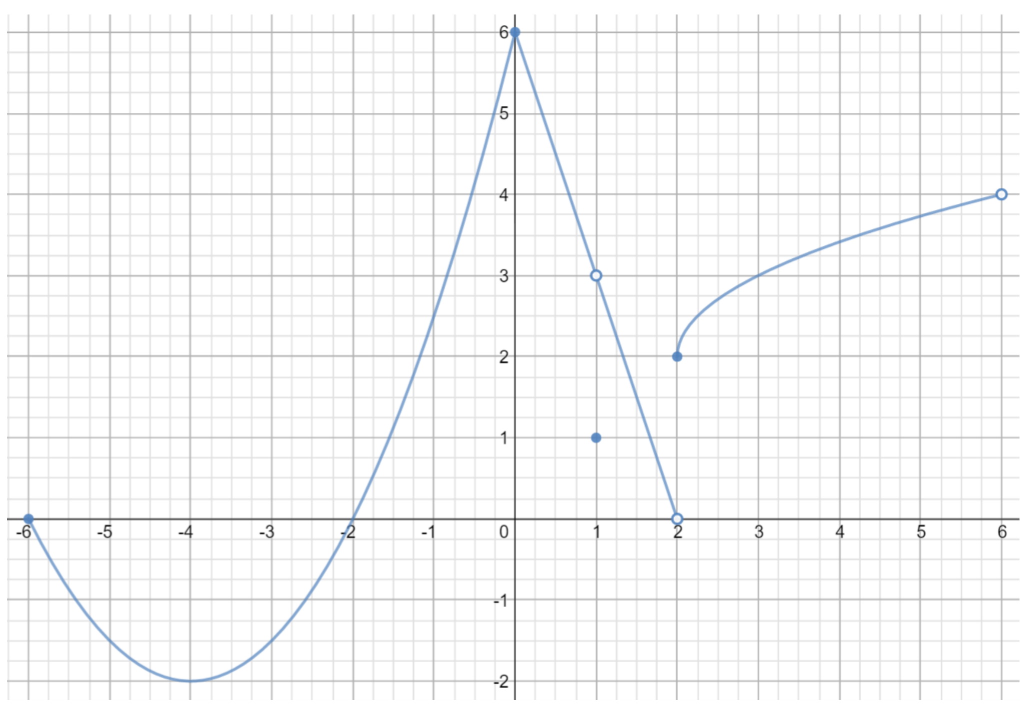
D) $\lim_{x \to -6} f(x)$

E) $\lim_{x \to 1} f(x)$

F) $f(1)$

G) $f(2)$

H) $\lim_{x \to 2} f(x)$



Properties of Limits

Let b and c be real numbers and let f and g be functions with the following limits:

$$\lim_{x\to c} f(x) = L \text{ and } \lim_{x\to c} g(x) = K$$

Rule	Rule
$\lim_{x\to c} b = b$	$\lim_{x\to c}[b \cdot g(x)] = bL$
$\lim_{x\to c} x = c$	$\lim_{x\to c}[f(x)]^{\frac{r}{s}} = L^{\frac{r}{s}}$ provided r and s are integers and $s \neq 0$.
$\lim_{x\to c}[f(x) \pm g(x)] = L \pm K$	$\lim_{x\to c}\left[\frac{f(x)}{g(x)}\right] = \frac{L}{K}$ provided $K \neq 0$.
$\lim_{x\to c}[f(x) \cdot g(x)] = L \cdot K$	

Example 1: Use the given information to evaluate the limits: $\lim_{x\to c} f(x) = 2$ and $\lim_{x\to c} g(x) = 3$

a) $\lim_{x\to c}[5g(x)]$

b) $\lim_{x\to c}[f(x) + g(x)]$

c) $\lim_{x\to c}[f(x)g(x)]$

d) $\lim_{x\to c} \frac{f(x)}{g(x)}$

Finding Limits by Direct Substitution

When tasked with evaluating a limit, the first thing you should always try is direct substitution. Direct substitution means that you plug-in the value that x approaches and determine what the expression evaluates to. As long as the expression does not evaluate to an undefined value, direct substitution will work.

Example 2: Evaluate the following limits.

a) $\lim_{x\to 1}(-x^2 + 1)$

b) $\lim_{x\to 3} \frac{\sqrt{x+1}}{x-4}$

c) $\lim_{h\to 0}(3h^2 + 2h)$

d) $\lim_{h\to 0}(3x^2 - 2xh + 5h)$

AB Calculus: Finding Limits Algebraically **Name: ______________________________**

When evaluating a limit, the technique you always want to try first is direct substitution. If a limit cannot be found using direct substitution, then we will use other techniques to try and evaluate the limit. Keep in mind that some functions do not have limits.

> If direct substitution yields $\frac{0}{0}$, otherwise known as **indeterminate form**, then you cannot determine the limit in its current form. Encountering this form means you should try another technique. One way to deal with limits in this form is to use algebraic techniques like **factoring**, **simplifying**, and **rationalizing the numerator**.

Example 3: Evaluate the following limits.

a) $\lim\limits_{x\to -1} \dfrac{2x^2 - x - 3}{x + 1}$

b) $\lim\limits_{x\to 3} \dfrac{\sqrt{x+1} - 2}{x - 3}$

c) $\lim\limits_{x\to 0} \dfrac{\frac{1}{x+4} - \frac{1}{4}}{x}$

d) $\lim\limits_{x\to 2} \dfrac{3x^2 + 5x - 2}{x - 2}$

If direct substitution yields to a fraction with 0 in the denominator and something other than 0 in the numerator, then either you have a limit that increases or decreases without bound (∞ or $-\infty$), or the limit does not exist.

Example 4: Investigate $\lim\limits_{x\to 0} \frac{\sin x}{x}$ by looking at the graph and making a table.

x	-0.1	-.01	-.001	0	.001	.01	.1
$f(x)$							

Based on your investigation, what do you think the value of the limit is?

You must understand that while using a graph and/or table allows us to estimate the value of a limit, we have not proved it is the value until we algebraically confirm the limit is what we think it is. The proof of the limit in example 4 is a little more complicated, but for now, you will want to memorize this limit.

Example 5: Evaluate the following limits.

a) $\lim_{k\to 0} \frac{\sin k}{k}$ b) $\lim_{x\to 0} \frac{\sin 5x}{4x}$ c) $\lim_{x\to 0} \frac{\sin x}{5x^2 + x}$

The Sandwich Theorem (aka The Squeeze Theorem)

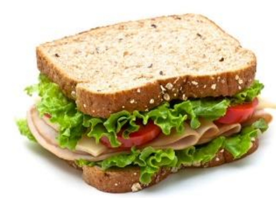
The Sandwich Theorem

If $g(x) \le f(x) \le h(x)$ for all $x \ne c$ in some interval about c, and

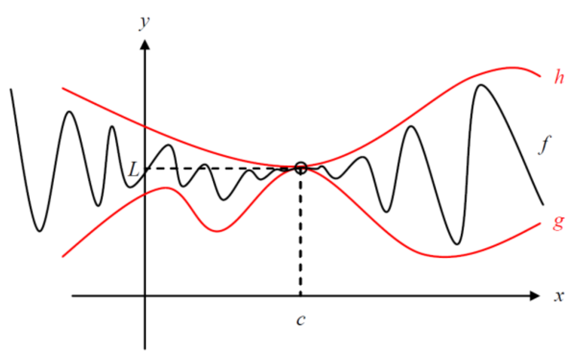
$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L,$$

then

$$\lim_{x\to c} f(x) = L$$



In other words, if we "sandwich" the function $f(x)$ between two other functions $g(x)$ and $h(x)$ that both have the same limit as x approaches c, then $f(x)$ is forced to have the same limit too.



Example 6: Use the Sandwich Theorem to evaluate the following limits.

a) If $5 - 3x - x^2 \le g(x) \le x + 9$ find $\lim_{x\to -2} g(x)$

b) $\lim_{x\to 0} x^2 \cos\left(\frac{1}{x}\right)$

AB Calculus: Limits Involving Infinity

We are going to look at two kinds of limits involving infinity. The first type is determining what happens to a function as x approaches infinity in either the positive or negative direction ($x \to \pm\infty$). The second type is functions whose limit approaches infinity in either the positive and negative direction as x approaches a given value.

The first type: Limits as $x \to \pm\infty$

Example 1: Use your calculator or Desmos to investigate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for the following functions:

a) $f(x) = \dfrac{1}{x}$

b) $f(x) = \dfrac{2x-1}{x+3}$

Definition: Horizontal Asymptote

The line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \text{ or } \lim_{x\to-\infty} f(x) = b$$

The horizontal asymptotes of rational functions (the quotient of two polynomials like in example 1) will be the same in both the positive and negative directions. When dealing with rational functions, you can find horizontal asymptotes using the same rules you learned in previous math classes.

Using Rules for Finding Horizontal Asymptotes to find Infinite Limits

Relationship	Asymptote	Example
Degree of Numerator is Larger		
Degree of Denominator is Larger		
Degrees are the Same		

Another technique that can be used to find the horizontal asymptotes of a function is to use an end behavior model (EBM). An EBM is a function that is simpler than the original, but behaves in the exact same way as the original as x gets really big in either the positive or negative direction.

Finding Horizontal Asymptotes Using End Behavior Models

End Behavior Model for Rational Functions

For a rational function $\frac{ax^m + \dots}{bx^n + \dots}$ where m is the degree of the numerator and n is the degree of the denominator, the end behavior model, or the function that the original behaves like when x gets sufficiently large in the negative or positive direction, can be written as

$$\frac{ax^m}{bx^n} \text{ or } \frac{a}{b}x^{m-n}$$

Example 2: Find the end behavior model (EBM) for the following functions.

a) $\lim_{x\to\infty} \frac{3x^4 - 2x^3 + 3x^2 - 5x + 6}{5x^4}$

b) $\lim_{x\to\infty} \frac{2x^5 + x^4 - x^2 + 1}{3x^2 - 5x + 7}$

c) $\lim_{x\to\infty} \frac{2x^3 - x^2 + x - 1}{5x^2 + x^3 + x - 5}$

d) $\lim_{x\to\infty} \frac{x^2 + x - 1}{x^4 - 3x^2 + 2x - 6}$

Example 3: For each of the following, find the end behavior model, evaluate the limit, and find any horizontal asymptotes.

a) $\lim_{x\to\infty} \frac{2x + 5}{3x^2 - 6x + 1}$

b) $\lim_{x\to\infty} \frac{2x^2 - 3x + 5}{x^2 + 1}$

c) $\lim_{x\to-\infty} \frac{x^4 + x^3 + 9}{3x - 3}$

Horizontal Asymptotes of Non-Rational Functions

In previous examples, the horizontal asymptotes in both the positive and negative direction were the same. Once again, this occurs whenever you have a rational function. This is not necessarily the case for other functions that are created from the quotient of two functions. There are even functions that have more than one horizontal asymptote! As a general rule for functions that are divided, if the denominator "grows" faster that the numerator in the given direction, the limit as x approaches infinity will be 0; if the numerator "grows" faster than the denominator, the limit will increase or decrease without bound; if they grow at the same rate, then the limit will be a constant value.

Growth Rate
Factorial Exponential Polynomial Logarithmic sin x/cos x constants

Example 4: Evaluate the following limits.

a) $\lim_{x\to\infty} \frac{\sin x}{x}$

b) $\lim_{x\to-\infty} e^x - 2x$

c) $\lim_{x\to\infty} \frac{5+2^x}{2-2^x}$

d) $\lim_{x\to-\infty} \frac{5+2^x}{2-2^x}$

The Second Type: Infinite Limits as $x \to a$

Example 5: Investigate the following limits.

a) $\lim_{x\to 0^-} \frac{1}{x}$

b) $\lim_{x\to 0^+} \frac{1}{x}$

Definition: Vertical Asymptote

The line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^-} f(x) = \pm\infty \text{ or } \lim_{x\to a^+} f(x) = \pm\infty$$

Note: This occurs whenever there is a value of x that gives you a 0 in the denominator but not the numerator. If the value of x gives you a 0 in both the numerator and the denominator, the graph has a hole.

Example 6: Find the vertical asymptotes of each function, find the limit of the function as x approaches from the left and the right of each asymptote, then find the value of the limit at each asymptote.

a) $f(x) = \dfrac{x^2 - 1}{2x - 4}$

b) $f(x) = \dfrac{1 - x}{x^2 - 4x + 3}$

c) $f(x) = \dfrac{1}{(x + 1)^2}$

In general, if the limit increases without bound from both the left and the right of the asymptote, the value of the limit is ∞. If the limit decreases without bound from both the left and the right, the limit is $-\infty$. If the limit from the left and right are not the same, we say the limit does not exist (DNE). Remember, a limit value of ∞ or $-\infty$ technically means the limit does not exist (it is one of the three common behaviors discussed previously that caused limits not to exist). However, we give the more specific answer of ∞ or $-\infty$ because it distinguishes between limits that behave the same way from both sides of the value, and limits that do not. From this day forward, DNE is not an acceptable answer for a limit value that should be ∞ or $-\infty$.

Graphs and Limits involving infinity

Example 8: Use the graph to evaluate the following limits

A $f(0)$

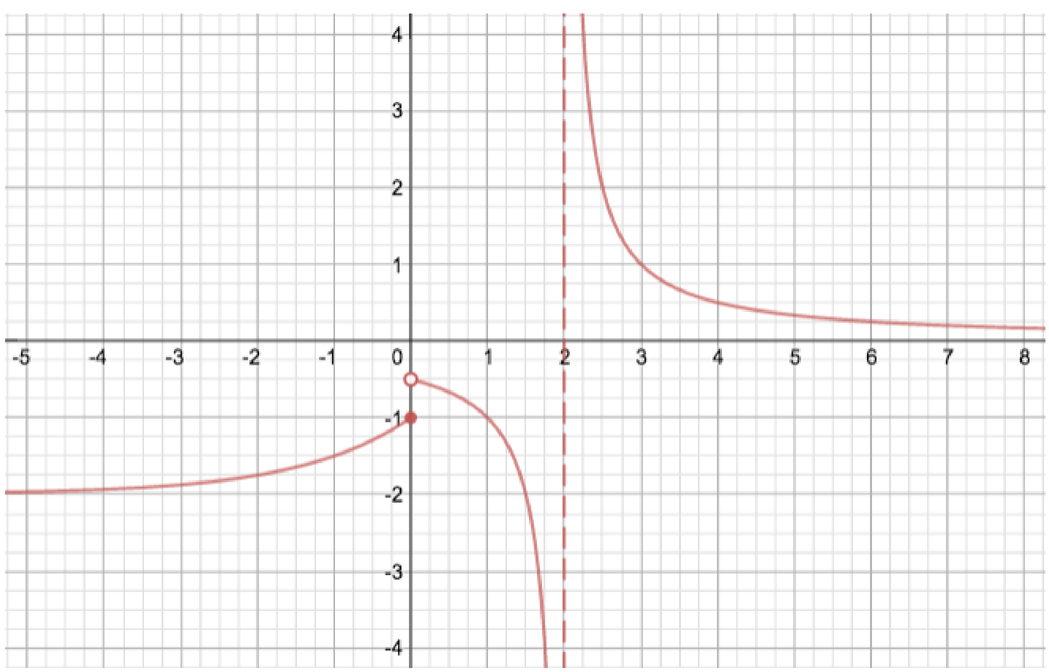
B $\lim\limits_{x \to -\infty} f(x)$

C $\lim\limits_{x \to \infty} f(x)$

D $\lim\limits_{x \to 2^+} f(x)$

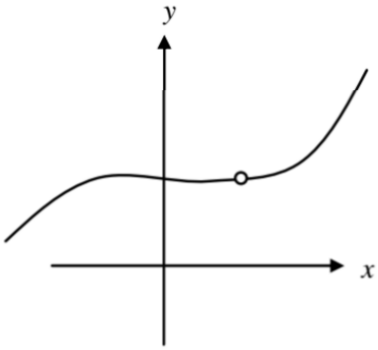
E $\lim\limits_{x \to 2^-} f(x)$

F $f(2)$

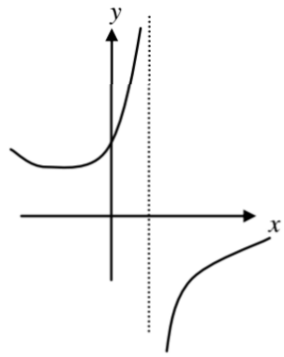


AB Calculus: Continuity

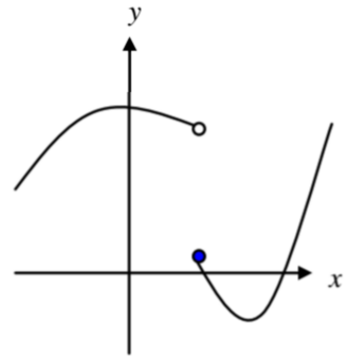
A function is continuous if you can draw the function without ever lifting your pencil. The following graphs demonstrate three types of discontinuous graphs.



Removable Discontinuity
Hole in the graph



Non-Removable Discontinuity
Vertical Asymptote



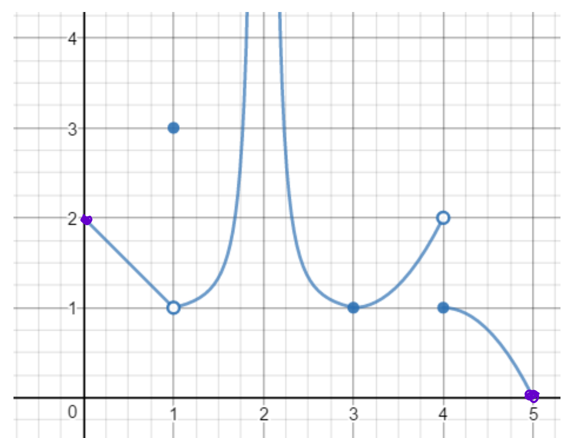
Non-Removable Discontinuity
Jump

There are two types of discontinuities, removable and non-removable. A hole in the graph is an example of a removable discontinuity. It is considered removable because you can easily make the graph continuous again by filling the hole. Vertical asymptotes and jumps are examples of non-removable discontinuities. They cannot be made continuous without drastically changing the function itself.

Example 1: Find the points (intervals) at which the function below is continuous, and the points at which it is discontinuous over the interval $0 < x < 5$.

Cont
$[0,1)$
$(1,2)$
$(2,4)$
$[4,5]$

POD
$x=1$ Hole
$x=2$ VA
$x=4$ Jump



Domain $[0,2)(2,5]$

Continuity at a point

A function $y = f(x)$ is continous at point c if

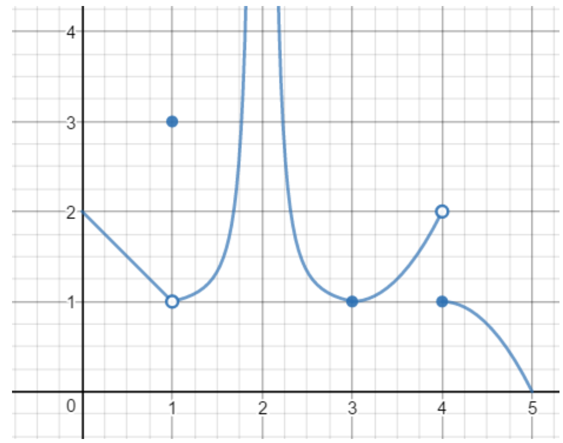
$$\boxed{\lim_{x \to c} f(x) = f(c)}$$

Or in other words,

$$\boxed{\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c)}$$

"The limit from the left, equals the limit from the right, equals the function value."

Example 2: For $c = 1$, 2, and 4, find $f(c)$, $\lim_{x\to c^+} f(x)$, $\lim_{x\to c^-} f(x)$, and $\lim_{x\to c} f(x)$. Take notice how each part of the definition of continuity is important.



$\lim_{x\to 1^+} f(x) = 1$

$\lim_{x\to 1^-} f(x) = 1$

$\lim_{x\to 1} f(x) = 1$

$f(1) = 3$

Not cont b/c $\lim_{x\to 1} f(x) \neq f(1)$

$f(2)$ is undefined

$\therefore$ not continuous

$\lim_{x\to 4^+} f(x) = 1$

$\lim_{x\to 4^-} f(x) = 2$

$\therefore$ $\lim_{x\to 4} f(x)$ DNE

You don't always get a picture, so you will have to be able to do this algebraically as well.

Example 3: Determine whether each function is continuous or not. If it is not continuous, use the definition of continuity to explain why.

a) $f(x) = \dfrac{1}{x-1}$ VA at $x=1$ $f(1)$ is undefined

$x+2=0$
$-2\ \ -2$
$x=-2$

b) $g(x) = \dfrac{2x^2+x-6}{x+2}$ $\rightarrow$ $\dfrac{(2x-3)\cancel{(x+2)}}{\cancel{x+2}}$ Hole at $y=-2$ $f(-2)$ is undefined

c) $h(x) = \begin{cases} -2x+3, & x < 1 \\ x^2, & x \geq 1 \end{cases}$

$\lim_{x\to 1^-} -2x+3 = 1$

$\lim_{x\to 1^+} x^2 = 1$

$\therefore$ $\lim_{x\to 1} h(x) = 1$

$f(1) = 1^2 = 1$

Example 4: Use the definition of continuity to find the value of a so that g(x) will be continuous for all real numbers.

a) $g(x) = \begin{cases} x^2+7, & x \geq 1 \\ x+a, & x < 1 \end{cases}$

$x^2+7 = x+a$

$1^2+7 = 1+a$

$a=7$

b) $h(x) = \begin{cases} \dfrac{x^4-1}{x-1}, & x \neq 1 \\ k, & x = 1 \end{cases}$

$\dfrac{1^4-1}{1-1} = \dfrac{0}{0}$ IND

$\dfrac{(x^2+1)(x^2-1)}{x-1} = \dfrac{(x^2+1)(x+1)\cancel{(x-1)}}{\cancel{x-1}}$

$(1^2+1)(1+1) = k$

$2\ \ 2 = k$

so $\boxed{k=4}$

Properties of Continuity

If b is a real number and f and g are continuous at $x = c$, then the following functions are continuous at c.

1. Constant multiple: $b \cdot f$
2. Sum and Difference: $f \pm g$
3. Product: $f \cdot g$
4. Quotient: $\frac{f}{g}$; $g(c) \neq 0$

Extended Functions

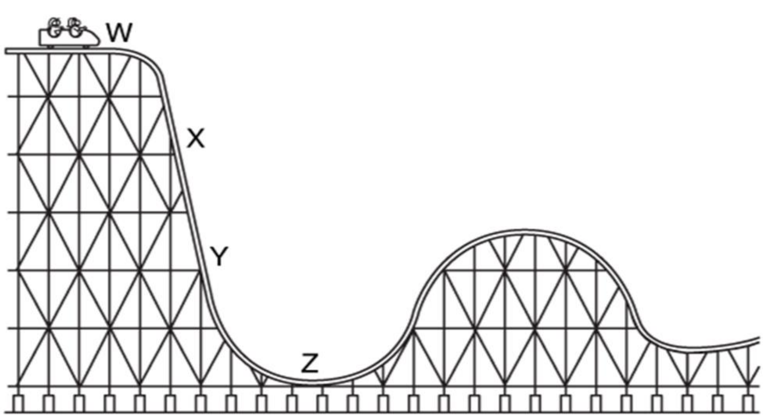
An extended function is a function that is obtained after a discontinuity is removed. To write an extended function, find where the hole in the graph, then write a piecewise function to fill the hole.

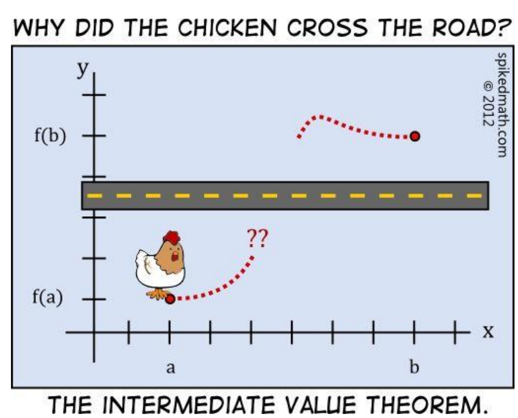
Example 4: Write an extended function to remove the removable discontinuity from the function $f(x)$.

a) $f(x) = \dfrac{x^2 - 6x + 5}{x - 1} = \dfrac{(x-5)(x-1)}{x-1}$ $\quad f(x) = \begin{cases} x-5 & x \neq 1 \\ -4 & x = 1 \end{cases}$

b) $f(x) = \dfrac{x^2 - 3x - 18}{x + 3} = \dfrac{(x-6)(x+3)}{x+3}$ $\quad f(x) = \begin{cases} x-6 & x \neq -3 \\ -9 & x = -3 \end{cases}$

Intermediate Value Theorem (IVT)





For the car to go from point W to point Z safely, do you have to go through points X and Y? Why or Why not?

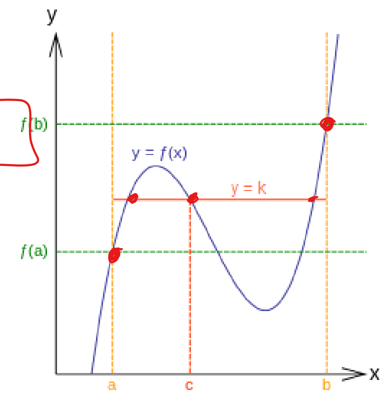
yes Stay on track

Intermediate Value Theorem (IVT)

If f is continuous on the closed interval $[a, b]$ then f takes every value between $f(a)$ and $f(b)$.

Suppose k is a value between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = k$.

The Intermediate Value Theorem tells you that at least one c exists, but it does not give you a method of finding c. This theorem is an example of an **existence theorem**.



Example 5: In the Intermediate Value Theorem …

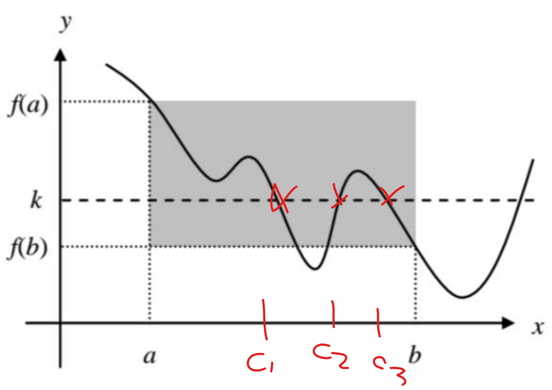
a) What are the necessary requirements in order to apply this theorem?

continuous on closed interval $[a,b]$

b) k is on which axis? y

c) c is on which axis? x

Example 6: Consider the function below and answer the questions.



a) Is f continuous on $[a, b]$? yes

b) Is k between $f(a)$ and $f(b)$? yes

c) In this example, if $a < c < b$, then there are 3 $c's$ such that $f(c) = k$.

d) Label the $c's$ on the graph as $c_1, c_2, \ldots$

Example 7: Verify that the Intermediate Value Theorem applies to the following function $f(x)$ over the interval $\left[\frac{5}{2}, 4\right]$, explain why IVT guarantees an x-value of c where $f(c) = 6$, and find c.

$$f(x) = \frac{x^2 + x}{x - 1}$$

$f(x)$ is continuous on $\left[\frac{5}{2}, 4\right]$

$$f\left(\frac{5}{2}\right) = \frac{\left(\frac{5}{2}\right)^2 + \frac{5}{2}\cdot\frac{2}{2}}{\frac{5}{2} - 1\cdot\frac{2}{2}} = \frac{\frac{25}{4} + \frac{10}{4}}{\frac{3}{2}} = \frac{35}{4}\cdot\frac{2}{3} = \frac{35}{6}$$

$$f(4) = \frac{4^2 + 4}{4 - 1} = \frac{16+4}{3} = \frac{20}{3}$$

$$\frac{35}{6} < 6 < \frac{20}{3}$$

The IVT states that there is a c in $\left[\frac{5}{2}, 4\right]$ such that $f(c) = 6$

$$6 = \frac{x^2 + x}{x - 1}$$

$$6(x-1) = x^2 + x$$

$$6x - 6 = x^2 + x$$

$$-6x + 6 \qquad -6x + 6$$

$$0 = x^2 - 5x + 6$$

$$0 = (x-3)(x-2)$$

$c = 3$

*2 not in interval $\left[\frac{5}{2}, 4\right]$