AP Calculus Linearization and Differentials Name: ___________________________

1. If $f(x) = x^3 - 2x + 3$,

   a) Find the linearization $L(x)$ of $f(x)$ centered at $x = 2$.

$f'(x) = 3x^2 - 2$

$f'(2) = 3(2)^2 - 2 = 10$

$f(2) = 2^3 - 2(2) + 3 = 7$

$y - 7 = 10(x - 2)$

$\boxed{L(x) \approx 10(x-2) + 7}$

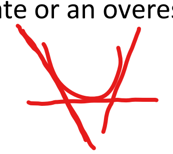
   b) Use $L(x)$ to approximate $f(2.1)$

$L(2.1) = 10(2.1 - 2) + 7 = 10(.1) + 7 = 8 \rightarrow f(2.1) \approx 8$

   c) Determine if the approximation is an underestimate or an overestimate.

$f''(x) = 6x$ $f''(2) = 12$

cup



underestimate

   d) Use a calculator to determine the accuracy of the approximation you found in part b.

$|f(2.1) - L(2.1)| \rightarrow 8.061 - 8 = .061$

2. If $f(x) = \sqrt{x^2 + 9}$, $f(x) = (x^2 + 9)^{\frac{1}{2}}$

   a) Find the linearization $L(x)$ of $f(x)$ centered at $x = -4$.

$f'(x) = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+9}}$ / $f'(-4) = \frac{-4}{\sqrt{16+9}} = -\frac{4}{5}$ / $f(-4) = \sqrt{16+9} = 5$

$y - 5 = -\frac{4}{5}(x + 4)$

$\boxed{L(x) = -\frac{4}{5}(x+4) + 5}$

   b) Use $L(x)$ to approximate $f(-3.9)$

$f(-3.9) \approx -\frac{4}{5}(-3.9 + 4) + 5 = -\frac{4}{5}(.1) + 5 = -.08 + 5 = \boxed{4.92}$

   c) Use a calculator to determine the accuracy of the approximation you found in part b.

Calc says $f(-3.9) = 4.920365$ error $= .000365\ldots$

3. Approximate the value of $\sqrt{101}$ by using a linearization of a nearby number.

$f(x) = \sqrt{x}$ center at $x = 100$

$f(100) = \sqrt{100} = 10$

$f'(x) = \frac{1}{2\sqrt{x}}$ $f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$

$y - 10 = \frac{1}{20}(x - 100)$ / $L(x) = \frac{1}{20}(x - 100) + 10$

$f(101) \approx$

$\frac{1}{20}(101 - 100) + 10$

$\frac{1}{20} + 10$

$\boxed{= 10.05}$

4. For each of the following, find dy and evaluate dy for the given value of x and dx.

a) $y = x^3 - 3x, x = 2, dx = 0.05$

$\frac{dy}{dx} = 3x^2 - 3$ / $dy = (3x^2 - 3)\,dx$

$dy\big|_{x=2,\,dx=.05} = (3(2)^2 - 3)(.05) = .45$

b) $y = x^2 \ln x, x = 1, dx = 0.01$

$\frac{dy}{dx} = 2x\ln x + x^2 \frac{1}{x} \rightarrow$ simplifies to x

$dy = (2x\ln x + x)\,dx \rightarrow dy\big|_{x=1,\,dx=.01} = (2(1)\ln 1 + 1)(.01) = .01$

5. Use differentials to estimate $\underbrace{\text{the change in the volume}}_{dV}$ of a sphere $\left(V = \frac{4}{3}\pi r^3\right)$ when the radius changes from 10 cm to 10.05 cm.

$dr = .05$

$\frac{dV}{dr} = 3\left(\frac{4}{3}\right)\pi r^2 \rightarrow dV = 4\pi r^2\,dr \rightarrow dV\big|_{r=10,\,dr=.05} = 4\pi(10)^2(.05) = \boxed{20\pi\text{ cm}^3}$

6. Evaluate the following limits.

a) $\lim_{x\to 0^+} x \ln x$

$\lim_{x\to 0^+} \frac{\ln x}{1/x}$

$\lim_{x\to 0^+} \ln x \rightarrow -\infty$

$\lim_{x\to 0^+} \frac{1}{x} \rightarrow \infty$

$\lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\cancel{\frac{-1}{x^2}}}$ $\frac{x^2}{-1}$ $\rightarrow -x$ $\boxed{=0}$

b) $\lim_{x\to\frac{\pi}{2}^+} \frac{1+\sec x}{\tan x}$

$\lim_{x\to\frac{\pi}{2}^+} (1+\sec x) \rightarrow -\infty$

$\lim_{x\to\frac{\pi}{2}^+} \tan x \rightarrow -\infty$

$\lim_{x\to\frac{\pi}{2}^+} \frac{1+\sec x}{\tan x} = \lim_{x\to\frac{\pi}{2}^+} \frac{\cancel{\sec} x\tan x}{\sec^{\cancel{2}} x}$

$\frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}} = \sin x$

$\sin\frac{\pi}{2} = 1$

c) $\lim_{x\to 0}(e^x + x)^{\frac{1}{x}}$

$\lim_{x\to 0} \frac{1}{x}\ln(e^x + x)$

$\lim_{x\to 0} \frac{\ln(e^x+x)}{x}$

$\lim_{x\to 0} \ln(e^x+x) = 0$

$\lim_{x\to 0} x = 0$

$\lim_{x\to 0} \frac{\ln(e^x+x)}{x} = \lim_{x\to 0} \frac{\frac{1}{e^x+x}(e^x+1)}{1}$

$\downarrow$

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$\boxed{\text{Ans } e^2}$