AB Calculus: Related Rates **Name:** ______________________________

Earlier in the year, we used the basic definition of calculus as "the study of change." Anytime we use any words such as increasing, decreasing, growing, or shrinking, we are talking about calculus. Change occurs over time, so when we talk about how a quantity changes, we are talking about the derivative of that quantity with respect to time. We can state this mathematically as $\frac{d}{dt}[\text{quantity}]$.

Example 1 Write the following statements mathematically

a) Mortimer is growing at a rate of 3 inches per year.

↳ g $\quad \frac{dg}{dt} = 3''/\text{year}$

b) My stock portfolio is losing 5 cents per day

→ P $\quad \frac{dP}{dt} = -5$ ¢/day

c) The radius of a circle gets larger by 4 feet each hour

$\frac{dr}{dt} = 4$ ft/hr

d) The outside temperature is dropping by 5 degrees Fahrenheit per minute

$\frac{dT}{dt} = -5$ °F/min

There are two keys to solving related rates problems. First is understanding that you are always taking the derivative with respect to time, which we will represent using t. Therefore, if we take the derivative of a variable that is not a t, then we assume that the variable can be written as a function of t, so, just like implicit differentiation, we use the chain rule.

Example 2 Take the derivative of the following functions with respect to time.

a) $a^2 + b^2 = c^2$

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

b) $A = \pi r^2$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

c) $\tan\theta = \frac{x}{5}$ → $\tan\theta = \frac{1}{5}x$

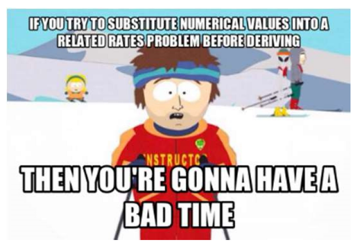
$$\sec^2\theta\frac{d\theta}{dt} = \frac{1}{5}\frac{dx}{dt}$$

d) $x = \ln(2t)$

$$\frac{dx}{dt} = \frac{1}{2t}(2) = \frac{1}{t}\cdot\cancel{\frac{dt}{dt}}$$

implied to already $=1$

The second key to related rates is understanding when you can substitute numerical values. If the value represents a rate, then you have to wait to substitute until you derive. For other values, you have to decide whether the value changes throughout the problem or not. If the numerical value would not change throughout the course of the scenario, then we can substitute it into the equation in the beginning. If the value does change, then we are talking about a value at an instant and we cannot substitute until after we take the derivative.



Example 3 Determine which numerical values can be substituted before deriving and which cannot.

a) Assume that oil spilled from a ruptured tank spreads in a circular pattern whose radius increases at a constant rate of 2 ft/sec. How fast is the area of the spill increasing when the radius of the spill is 60 ft?

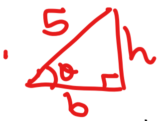
$\frac{dr}{dt} = 2$ ft/sec

the ?? is trying to find what?

$\frac{dA}{dt}$

r=60 ft
↓
can't plug in until after we derive

b) A 5-ft. ladder, leaning against a wall, slips so that its base moves away from the wall at a rate of 2 ft/sec. How fast will the top of the ladder be moving down the wall when the base is 4 ft. from the wall.

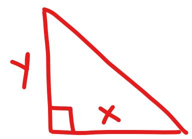


$\frac{db}{dt} = 2$ $\boxed{\frac{dh}{dt} = ?}$ $b = 4 \rightarrow$ wait to plugin until we derive

c) A right triangle whose sides are changing has sides of 30 and 40 inches at a particular instant. If the shorter of these two sides is increasing at 3 in/sec and the longer side is decreasing at 5 in/sec, how fast is the area changing?

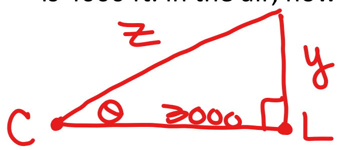
$\frac{dA}{dt} = ?$

$x = 30$
$y = 40 \rightarrow$ wait



$\frac{dx}{dt} = 3$ $\frac{dy}{dt} = -5$

d) A camera is placed 3000 ft. from a rocket launching pad. If a rocket is rising vertically at 880 ft/sec when it is 4000 ft. in the air, how fast is the camera-to-rocket distance changing at that instant?



$\frac{dy}{dt} = 880$ ft/sec (only when $y = 4000$)

Asking for $\frac{dz}{dt}$

To solve related rates problems, you need a strategy that always works. Related rates problems always can be recognized by the words increasing, decreasing, growing, shrinking, changing. Follow these steps when solving related rates problems.

1. Draw a picture. Label all sides and quantities that change throughout the life of the problem as convenient variables of your choice. If something in the problem is not changing throughout the problem, label it with its constant value.
2. State what you are given in terms of a rate.
3. State what you are trying to find (Trying to find ____ when ____)
4. Before you can relate your rates to each other, you need to relate their quantities. Find an equation which ties your variables together. If it is an area problem, you need an area equation. If it is a right triangle, the Pythagorean Theorem formula may work or general trigonometry formulas may apply.
5. If your equation is a function of two variables, write a secondary equation and solve for one of the variables to get your related rates equation in terms of a single variable.
5. Use implicit differentiation to take the derivative of both sides with respect to t (time).
6. Substitute in known information and solve for the value you are trying to find.
7. Answer the question with a sentence summarizing your conclusion with appropriate units. Make sure to reference the specific moment in time that you found the value for.
8. Remember, you are awesome!

Example 4 Assume that oil spilled from a ruptured tank spreads in a circular pattern whose radius increases at a constant rate of 2 ft/sec. How fast is the area of the spill increasing when the radius of the spill is 60 ft?

$\frac{dr}{dt} = 2$ $\frac{dA}{dt}$? when $r = 60$

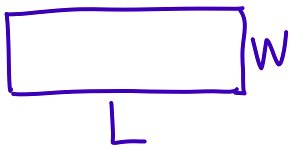
$\frac{d}{dt}\left[A = \pi r^2\right]$

$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$

$\left.\frac{dA}{dt}\right|_{r=60,\ \frac{dr}{dt}=2} = 2\pi(60)(2)$

$= 240\pi$ ft^2/sec

Example 5 The length of a rectangle is decreasing by 2 inches per second and the width is increasing by 3 inches per second. When the length is 10 inches and the width is 6 inches, how fast is the area changing?



$\frac{dL}{dt} = -2$ $\quad L = 10,\ W = 6$ $\quad \boxed{\frac{dA}{dt}}$

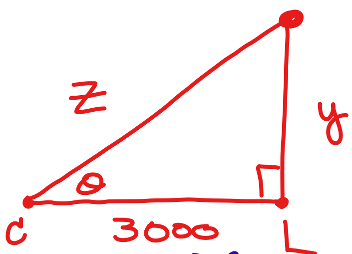
$\frac{dW}{dt} = 3$

$A = LW$

$\frac{dA}{dt} = \frac{dL}{dt} W + L \frac{dW}{dt}$

$\left.\frac{dA}{dt}\right|_{\frac{dL}{dt}, \frac{dW}{dt}, \text{etc}} = (-2)(6) + (10)(3) = -12 + 30 = 18\ in^2/sec$

Example 6 A camera is placed 3000 ft. from a rocket launching pad. If a rocket is rising vertically at 880 ft/sec when it is 4000 ft. in the air, how fast is the camera-to-rocket distance changing at that instant?



$\frac{dy}{dt} = 880$ when $y = 4000$ $\quad \frac{d}{dt}\left[3000^2 + y^2 = z^2\right]$

$\boxed{\frac{dz}{dt} = ?}$

$\cancel{2}y \frac{dy}{dt} = \cancel{2}z \frac{dz}{dt}$

$3000^2 + 4000^2 = z^2$

$z = 5{,}000$

$4{,}\cancel{000}(880) = (5{,}\cancel{000}) \frac{dz}{dt}$

$\frac{4(880)}{5} = \boxed{\frac{dz}{dt} = 704\ ft/sec}$

$5\overline{)880} = 176$ $\quad 176 \times 4 = 704$

Example 7 A camera is placed 3000 ft. from a rocket launching pad. If a rocket is rising vertically at 880 ft/sec when it is 4000 ft. in the air, how fast must the elevation angle change at that instant to keep the rocket in sight?

$\frac{d\theta}{dt} = ?$ $\quad \tan\theta = \frac{y}{3{,}000}$ $\quad \tan\theta = \frac{1}{3{,}000} y$

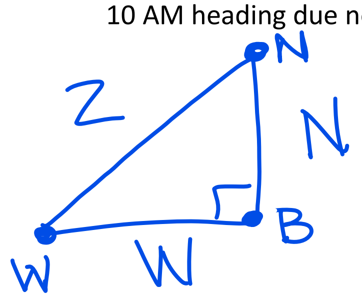
$\sec^2\theta \frac{d\theta}{dt} = \frac{1}{3{,}000} \frac{dy}{dt}$

$\frac{d\theta}{dt} = \frac{1}{3{,}000}(880) \frac{1}{\sec^2\theta} \rightarrow \frac{1}{3000}(880) \cos^2\theta$

$\frac{1}{3000}(880)\left(\frac{3000}{5000}\right)^2 \rightarrow \frac{88}{300} \cdot \frac{9}{25} = \boxed{\frac{66}{625}\ rad/sec}$

$88 = 4 \cdot 22 \cdot 2$ $\quad 9 = 3 \cdot 3$ $\quad 300 = 3 \cdot 100 = 50$ $\quad 25 \cdot 2$

Example 8 A boat passes a fixed buoy at 9 AM heading due west at 3 mph. Another boat passes the buoy at 10 AM heading due north at 5 mph. How fast is the distance between the boats changing at 11:30 AM?



$W^2 + N^2 = z^2$ $\quad 7.5^2 + 7.5^2 = z^2 \rightarrow z = 7.5\sqrt{2}$

$2W \frac{dW}{dt} + 2N \frac{dN}{dt} = 2z \cdot \frac{dz}{dt}$

$(7.5)(3) + (7.5)(5) = (7.5\sqrt{2}) \frac{dz}{dt}$

$8 = \sqrt{2} \frac{dz}{dt} \rightarrow \boxed{\frac{dz}{dt} = \frac{8}{\sqrt{2}}\ mph}$

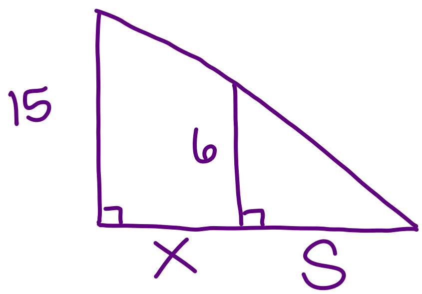
9 am – 11:30 am = 2.5 hr

$3 \cdot 2.5 = 7.5$

10 am – 11:30 = 1.5 hr

$5 \cdot 1.5 = 7.5$

Example 9 A street light is mounted at the top of a 15 ft. pole. A 6 ft. tall man walks away from the pole at a rate of 5 ft/sec. How fast is the length of his shadow moving when he is 40 ft. from the pole?



$$\frac{6}{s} = \frac{15}{x+s}$$

$$6(x+s) = 15s$$

$$6x + 6s = 15s$$

$$6x = 9s$$

$$2x = 3s \rightarrow 2\frac{dx}{dt} = 3\frac{ds}{dt}$$

$$2(5) = 3\frac{ds}{dt}$$

$$\boxed{\frac{10}{3} = \frac{ds}{dt} \text{ ft/s}}$$

Example 10 A spherical balloon is being inflated at a rate of 5 cubic inches per minute. When the radius of the balloon is 4 inches, how fast is the surface area of the balloon changing?

$\frac{dV}{dt} = 5$ when $r = 4$ $\frac{dA}{dt}$?

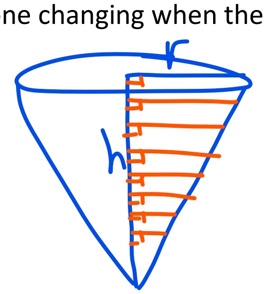
$$A = 4\pi r^2 \rightarrow \frac{dA}{dt} = 8\pi r \frac{dr}{dt} \rightarrow 8\pi(4)\frac{5}{64\pi} = \frac{32 \cdot 5}{64} = \boxed{\frac{5}{2} \text{ in}^2/\text{min}}$$

$$V = \frac{4}{3}\pi r^3 \rightarrow \frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt} \rightarrow 5 = 4\pi(4)^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{5}{64\pi}$$

Example 11 A rainbow snow cone is leaking from its paper cone at a rate of 2 cubic inches per minute. The paper cone's top radius is 3 inches and the paper cone is 5 inches tall. How fast is the radius of the snow cone changing when the radius of the snow cone is 2 inches?

$$\frac{dV}{dt} = -2$$



$r = 3$
$h = 5$

$$\frac{r}{h} = \frac{3}{5}$$

$$5r = 3h = h = \frac{5r}{3}$$

$$5\frac{dr}{dt} = 3\frac{dh}{dt}$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi r^2\left(\frac{5r}{3}\right)$$

$$V = \frac{5\pi}{9} r^3 \rightarrow \frac{dV}{dt} = \frac{15\pi}{9} r^2 \frac{dr}{dt}$$

$$-2 = \frac{15\pi}{9}(2)^2 \cdot \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{-2}{15\pi\, 4}\, 9$$

$$\boxed{= \frac{-3}{10\pi} \text{ in/min}}$$