AB Calculus U-Substitution Day 1 Notesheet **Name:** ______________________________

$x^3 \to 3x^2$

$\cos x \to -\sin x$

$\sqrt{x} \to \frac{1}{2\sqrt{x}}$

$x^2+5x \to 2x+5$

To Begin: Answer all parts of the following three questions.

1. Find the derivative of the following

a) $f(x) = \cos^3\sqrt{x^2+5x}$

$f'(x) = 3\left(\cos\sqrt{x^2+5x}\right)^2 \cdot -\sin\left(\sqrt{x^2+5x}\right) \cdot \frac{1}{2\sqrt{x^2+5x}} \cdot (2x+5)$

b) $g(x) = \int_4^{\cos x}(e^t+2t)dt$

$g'(x) = \left(e^{\cos x}+2\cos x\right)(-\sin x)$

2. Evaluate the following integrals.

a) $\int_{-1}^{2}(x^3-2x)dx$ $\boxed{\frac{3}{4}}$

$\left[\frac{1}{4}x^4 - x^2\right]_{-1}^{2}$

$=\left(\frac{1}{4}(2)^4-(2)^2\right)-\left(\frac{1}{4}(-1)^4-(-1)^2\right)$

b) $\int_1^2 \frac{3}{t^4}dt$ $\int_1^2 3t^{-4}\,dt$

$\left[\frac{3}{-3}t^{-3}\right]_1^2 = -1(2)^{-3}+1(1)^{-3}$

$= -\frac{1}{8}+1 = \boxed{7/8}$

c) $\int\left(\sec x\tan x + 2^x + \frac{1}{2x} + \frac{1}{\sqrt{1-x^2}}\right)dx$

$\sec x + \frac{1}{\ln 2}2^x + \frac{1}{2}\ln|x| + \sin^{-1}x + C$

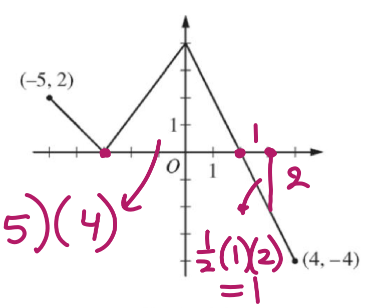
d) $\int\frac{x-1}{\sqrt{x}}dx = \int\left(\frac{x}{\sqrt{x}} - \frac{1}{\sqrt{x}}\right)dx$

$=\int x^{\frac{1}{2}} - x^{-\frac{1}{2}}$

$= \frac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + C$

3. AB Calculus Free Response Question (2014 #3)

$-\frac{4}{2} = -2$



Graph of f

The function f is defined on the closed interval [-5, 4]. The graph of f consists of three line segments and is shown in the figure to the left. Let g be the function defined by $g(x) = \int_{-3}^{x} f(t)dt$

a) Find g(3). $\int_{-3}^{3} f(t)dt = 10 - 1 = 9$

b) On what open intervals contained in $-5 < x < 4$ is the graph of g both increasing and concave down? Justify your answer.

$g'(x) = f(x)$ Inc $(-5,-3)(-3,2)$

$g''(x) = f'(x)$ cc down $(-5,-3)$ $(0,4)$

$\uparrow (-5,-3)$ $g'(x) > 0$

$(0,2)$ $g''(x) < 0$

c) The function h is defined by $h(x) = \frac{g(x)}{5x}$. Find $h'(3)$.

$h'(x) = \frac{(5x)g'(x) - (g(x))(5)}{25x^2}$

$h'(3) = \frac{(5\cdot 3)(g'(3)) - (g(3))(5)}{25(3)^2} = \frac{(15)(-2)-(9)(5)}{25(3)^2}$

d) The function p is defined by $p(x) = f(x^2-x)$. Find the slope of the line tangent to the graph of p at the point where $x = -1$.

$p'(x) = f'(x^2-x)\,(2x-1)$

$p'(-1) = f'((-1)^2-(-1))\,(2(-1)-1)$

$= f'(2)(-3) = (-2)(-3) = 6$

Up to this point you have been finding antiderivatives of functions that followed directly from their derivative rules. Today we learn how to undo a function that used the chain rule to find the derivative. First, a quick review of the chain rule:

Chain Rule for Derivatives

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

"Take the derivative of the outside, inside stays the same, then multiply by the derivative of the inside."

The chain rule gives us a product of two factors: the outside derivative and the inside derivative. So, if a function has this form, then it has an antiderivative. So, the first thing we need to be able to do is to tell whether or not an antiderivative is going to undo the chain rule. There are a couple of things to keep in mind. First, if the antiderivative undoes the chain rule, it will be a product of functions (sometimes one of these will just be a constant). Second, the constants are not important, but the variable parts are. Let's look at some examples.

Ex 1: Tell whether or not each antiderivative is going to undo a chain rule.

a) $\int (x^2 - 1)^3 2x\, dx$

inside $= x^2 - 1$

$\frac{d}{dx}(x^2-1) = 2x =$ outside

b) $\int 3x^2\sqrt{x^3+2}\, dx$

$\frac{d}{dx}(x^3+2)$

$3x^2 \rightarrow$ outside

c) $\int x(x^3 - 5)\, dx$

$\frac{d}{dx}(x^3-5) = 3x^2$

$\downarrow$

not outside

If the antiderivative will undo a chain rule, then we can use a method called u-substitution to integrate.

The U-Substitution Method for Evaluating Integrals

1. Identify the inside function. Let that function be u.
2. Find du by taking the derivative of u. Don't forget the dx (or d whatever variable you started with).
3. Solve your du equation for dx (or d whatever variable you started with).
4. Substitute in u and du. Any extra "x" parts should divide out or we will have to account for them.
5. Move any multiplied constants to the front.
6. Integrate. Don't forget +C if evaluating an indefinite integral.
7. Substitute your original inside function back for u.

Note: The goal will be to substitute so that all values of the integrand with either u or du. Any extras must be accounted for or substitution will not work. You can always check by taking the derivative of your answer. You should get the same thing you started with.

Ex 2: Evaluate the following integrals.

a) $\int \underline{(x^2 - 1)}^3 \underline{2x\, dx}$

$u = x^2 - 1$

$\frac{du}{dx} = 2x$

$du = 2x \cdot dx$

$\int u^3\, du$

$\frac{1}{4}u^4 + c$

$\boxed{\frac{1}{4}(x^2-1)^4 + c}$

b) $\int \underline{3x^2}\sqrt{\underline{x^3+2}}\, \underline{dx}$

$u = x^3 + 2$

$\frac{du}{dx} = 3x^2$

$du = 3x^2 dx$

$\int \sqrt{u}\, du$

$\int u^{\frac{1}{2}} du$

$\frac{2}{3}u^{\frac{3}{2}} + c$

$\boxed{\frac{2}{3}(x^3+2)^{\frac{3}{2}} + c}$

c) $\int \sin 3x\, dx$

$u = 3x$

$\frac{du}{dx} = 3$

$du = 3\,dx$

$\frac{1}{3} du = dx$

$\frac{1}{3}\int \sin u\, du$

$\to -\frac{1}{3}\cos u + c$

$\boxed{-\frac{1}{3}\cos(3x) + c}$

d) $\int \frac{\ln x}{x} dx$

$u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x} dx$

$\int u\, du = \frac{1}{2}u^2 + c$

$\boxed{= \frac{1}{2}(\ln x)^2 + c}$

Indefinite Integrals need a + C at the end of every antiderivative. Definite integrals have limits. If you change the variables, the limits still refer to the original variable. How will you decide to deal with those limits? You have two choices:

1. Leave the limits in terms of the original variable and integrate like you did for the indefinite integrals. Once you have returned all the variables back to the original letter, you can plug in the upper and lower limits.

Ex 3: Evaluate the following integrals.

a) $\int_{-1}^{1} \frac{x}{(x^2+1)^2} dx$

$u = 1 + 1 = 2$

$u = 1 + 1 = 2$

$u = x^2 + 1$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

$\frac{1}{2} du = x\,dx$

$\frac{1}{2}\int_2^2 \frac{1}{u^2} du$

$\boxed{= 0}$

b) $\int_0^2 x^2 e^{x^3} dx$

$u = x^3$

$u = 2^3 = 8$

$u = 0^3 = 0$

$\frac{du}{dx} = 3x^2$

$\frac{1}{3} du = x^2 dx$

$\frac{1}{3}\int_0^8 e^u\, du = \frac{1}{3}e^u \Big|_0^8$

$\boxed{= \frac{1}{3}e^8 - \frac{1}{3}e^0}$

2. Using the rule for the change of variables, change the limits with the same rule. Then, you never have to return to the original variable (Recommended).

Ex 4: Evaluate the following integrals.

a) $\int_0^2 x\sqrt{x^2+1}\,dx$

$u = 2^2 + 1 = 5$

$u = 0^2 + 1 = 1$

$u = x^2 + 1$

$\frac{du}{dx} = 2x$

$\frac{1}{2} du = x\,dx$

$\frac{1}{2}\int_1^5 \sqrt{u}\, du$

$\frac{1}{2} \frac{2}{3} u^{\frac{3}{2}} \Big|_1^5$

$= \frac{1}{3}(5)^{\frac{3}{2}} - \frac{1}{3}(1)^{\frac{3}{2}}$

$= \frac{1}{3}5\sqrt{5} - \frac{1}{3}$ OR $\frac{1}{3}(5\sqrt{5} - 1)$

b) $\int_0^{\pi^2} \frac{\cos\sqrt{x}}{\sqrt{x}} dx$

$u = \sqrt{\pi^2} = \pi$ $\quad u = \sqrt{0} = 0$

$u = \sqrt{x}$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$2\,du = \frac{1}{\sqrt{x}} dx$

$2\int_0^{\pi} \cos u\, du$

$\downarrow$

$2\sin u \Big|_0^{\pi}$

$2\sin\pi - 2\sin 0$

$0 - 0 = \boxed{0}$

Ex 4: Evaluate the following integrals.

a) $\int \frac{\cos x}{\sin^2 x}\,dx$

$u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x\,dx$

$\int \frac{1}{u^2}\,du$

$\int u^{-2}\,du$

$\frac{1}{-1}u^{-1} + c$ $\boxed{= -1(\sin x)^{-1} + c}$

b) $\int e^{\tan x}\sec^2 x\,dx$

$\int e^u\,du$

$u = \tan x$

$\frac{du}{dx} = \sec^2 x$

$du = \sec^2 x\,dx$

$e^u + c$

$\boxed{e^{\tan x} + c}$

c) $\int \frac{e^x}{e^x + 1}\,dx$

$u = e^x + 1$

$\frac{du}{dx} = e^x$

$du = e^x\,dx$

$\int \frac{1}{u}\,du$

$\ln|u| + c$

$\boxed{\ln|e^x + 1| + c}$

d) $\int \cancel{x^3}(2 + x^4)^5\,\cancel{dx}$

$\frac{1}{4}\int u^5\,du$

$u = 2 + x^4$

$\frac{du}{dx} = 4x^3$

$\frac{1}{4}du = x^3\,dx$

$\frac{1}{4}\cdot\frac{1}{6}u^6 + c$

$\boxed{\frac{1}{24}(2+x^4)^6 + c}$

e) $\int \cancel{\csc^2 x}\sqrt{\cot x}\,\cancel{dx}$

$u = \cot x$

$\frac{du}{dx} = -\csc^2 x$

$\frac{1}{-1}du = \csc^2 x\,dx$

$-\int \sqrt{u}\,du$

$-\int u^{\frac{1}{2}}\,du$

$-\frac{2}{3}u^{\frac{3}{2}} + c = \boxed{-\frac{2}{3}(\cot x)^{\frac{3}{2}} + c}$

f) $\int_1^2 \frac{e^{\frac{1}{x}}}{x^2}\,dx$

$u = \frac{1}{x}$

$\frac{du}{dx} = \frac{-1}{x^2}$

$\frac{1}{-1}du = \frac{1}{x^2}dx$

$u = \frac{1}{2}$

$u = \frac{1}{1}$

$-\int_1^{1/2} e^u\,du$

$-e^u\Big|_1^{1/2}$

$= -e^{1/2} + e^1 = \boxed{-\sqrt{e} + e}$

g) $\int_0^{\pi} \sec^2\left(\frac{t}{4}\right)dt$

$u = \frac{1}{4}\pi$

$u = \frac{1}{4}(0)$

$u = \frac{1}{4}t$

$\frac{du}{dt} = \frac{1}{4}$

$4\,du = dt$

$4\int_0^{\pi/4} \sec^2 u\,du$

$4\tan u\Big|_0^{\pi/4} = 4\tan\frac{\pi}{4} - 4\tan 0 = 4$

h) $\int_0^{\frac{1}{2}} \frac{\sin^{-1} x}{\sqrt{1 - x^2}}\,dx$

$u = \sin^{-1} x$

$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$

$du = \frac{1}{\sqrt{1-x^2}}dx$

$u = \sin^{-1}\frac{1}{2} = \pi/6$

$u = \sin^{-1} 0 = 0$

$\int_0^{\pi/6} u\,du = \frac{1}{2}u^2\Big|_0^{\pi/6}$

$= \frac{1}{2}(\pi/6)^2 - \frac{1}{2}(0)^2 = \boxed{\pi^2/72}$