BC Calculus Sequence Convergence/Divergence Notesheet **Name:** Key

A sequence is a list of things generated by a rule

More formally, a sequence is a function whose domain is the set of positive integers, or natural numbers, n such that $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. The range of the function are called the terms in the sequence,

$$a_1, a_2, a_3, \ldots a_n$$

Where a_n is called the nth term (or rule of the sequence), and we denote the sequence by $\{a_n\}$.

$4, 8, 12, 16, \ldots$ $a_n = \{4n\}$ $a_1 = 4$ $a_n = a_{n-1} + 4$

The sequence can be expressed by either

1. An ample number of terms in the sequence, separated by commas
2. An recursive function including a first term and a rule using the previous term of the sequence.
3. A rule for the sequence given as an explicit function set off in curly braces.

Example 1

The sequence $2, 4, 6, 8, \ldots$ is the sequence of even numbers. Express the same sequence as a rule of a non-negative integer n. The sequence $1, 3, 5, 7, \ldots$ is the sequence of odd numbers. Express the same sequence as a rule of a non-negative integer n. How many in the list are needed to establish the "rule" in the absence of the explicitly-stated rule?

$2, 4, 6, 8, \ldots$

$a_1 = 2$

$a_n = \{2n\}$ or $a_n = a_{n-1} + 2$

$1, 3, 5, 7$

$a_1 = 1$

$a_n = \{2n - 1\}$ $a_n = a_{n-1} + 2$

Note: When given a sequence as a list, the first term is usually designated to be associated with $n = 1$. This is because we are using n as a counting number.

We will be primarily interested in what happens to the sequence for increasingly large values of n.

Example 2

If $a_n = \left\{\frac{4n}{3+2n}\right\}$, list out the first five terms, then estimate $\lim_{n\to\infty} a_n$.

$\frac{4}{5}, \frac{8}{7}, \frac{12}{9}, \frac{16}{11}, \frac{20}{13}$ $\lim_{n\to\infty} a_n = 2$

Possibilities for Sequences as $n \to \infty$

Let $\{a_n\}$ be a sequence of real numbers.

- If $\lim_{n\to\infty} a_n = \infty$, then $\{a_n\}$ diverges to infinity.
- If $\lim_{n\to\infty} a_n = -\infty$, then $\{a_n\}$ diverges to negative infinity
- If $\lim_{n\to\infty} a_n = c$, a finite real number, then $\{a_n\}$ converges to c.
- If $\lim_{n\to\infty} a_n$ oscillates between two fixed numbers, then $\{a_n\}$ diverges by oscillation

Factorial

$n!$ Is read as "n factorial." It is defined recursively as $n! = n(n-1)!$ or as

$$n! = n(n-1)! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

For example, $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

Example 3 Determine whether the following sequences converge or diverge

a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$

$\lim_{n\to\infty} a_n = 1$ converges to 1

b) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$

$\lim_{n\to\infty} a_n = 0$ converges to 0

c) $a_n = \{3 + (-1)^n\}$

$2, 4, 2, 4, \ldots$

diverges by oscillating

d) $a_n = \left\{\frac{n}{1-2n}\right\}$

$\lim_{n\to\infty} a_n = -\frac{1}{2}$ converges to $-\frac{1}{2}$

e) $a_n = \left\{\frac{\ln n}{n}\right\} \xrightarrow{L'H} \frac{1/n}{1}$

$\lim_{n\to\infty} a_n = 0$ converges to 0

f) $a_n = \left\{\frac{n!}{(n+2)!}\right\}$ $\frac{\cancel{n!}}{(n+2)(n+1)\cancel{(n!)}}$

$\lim_{n\to\infty} a_n = 0$ converges to 0

g) $a_n = \left\{\frac{2n!}{(n-1)!}\right\}$

$\frac{2n\cancel{(n-1)!}}{\cancel{(n-1)!}}$ $\lim_{n\to\infty} a_n \to \infty$ diverges

h) $a_n = \left\{\frac{n + (-1)^n}{n}\right\}$

$\frac{n}{n} + \frac{(-1)^n}{n}$

$1 + \frac{(-1)^n}{n} \to 0$

$\lim_{n\to\infty} a_n = 1$

i) $a_n = \left\{\frac{(-1)^n(n-1)}{n}\right\}$

$\lim_{n\to\infty} (-1)^n \cdot \lim_{n\to\infty} \frac{n-1}{n}$ $\to 1$

$-1, 1 \cdot 1$

diverges by oscillation

j) $a_n = \left\{\frac{2^n}{(n+1)!}\right\}$

$\lim_{n\to\infty} a_n = 0$

$\frac{2^1}{2 \cdot 1} = 1$

$\frac{2^2}{3 \cdot 2 \cdot 1} = \frac{4}{6}$

$\frac{2^3}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{8}{24}$

$\frac{2^4}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{16}{120}$

$\frac{\text{EXP}}{\text{FACT}} \to 0$

k) $a_n = \left\{\left(1 + \frac{1}{n}\right)^n\right\}$ $n \ln\left(1 + \frac{1}{n}\right)$

$\frac{\ln(1+\frac{1}{n})}{\frac{1}{n}}$

L'H. $\frac{\frac{1}{1+\frac{1}{n}} \cdot \frac{-1}{n^2}}{\frac{-1}{n^2}} \to 1$

undo ln e^1

l) $a_n = \left\{\frac{(2n)!}{(2n-2)!}\right\}$

$\frac{(2n)(2n-1)\cancel{(2n-2)!}}{\cancel{(2n-2)!}}$

∞ diverges

Sometimes, we have to write the rule of a sequence as a function of n from a pattern.

Example 4 Determine whether the following sequences converge or diverge

a) $3, 8, 13, 18, \ldots$

$a_n = 5n - 2$

$\lim_{n\to\infty} a_n = \infty$

diverges

b) $5, -15, 45, -135, \ldots$

$a_n = 5(-3)^{n-1}$

$\lim_{n\to\infty} a_n \to$ oscillating

diverges

c) $1, 4, 9, 16, 25, \ldots$

$+3\ +5\ +7\ +9$

$a_n = n^2$

$\lim_{n\to\infty} a_n \to \infty$

diverges

d) $4, 10, 28, 82, \ldots$

$a_n = (3)^n + 1$

$\lim_{n\to\infty} a_n \to \infty$

diverges

e) $\frac{2}{1}, \frac{3}{3}, \frac{4}{5}, \frac{5}{7}, \frac{6}{9}, \ldots$

$a_n = \frac{n+1}{2n-1}$

$\lim_{n\to\infty} a_n = \frac{1}{2}$

f) $\ln 1, \ln 2, \ln 4, \ln 8, \ldots$

$a_n = \ln(2^{n-1})$

or $\ln\left(\frac{2^n}{2}\right) \to \infty$

diverges

AP Calculus Sequences Homework Name: Key

1. Write the first five terms of the sequence.

a) $a_n = 3^n$ $3, 9, 27, 81, 243$

b) $a_n = \sin\left(\frac{n\pi}{2}\right)$ $\sin\frac{\pi}{2}, \sin\frac{2\pi}{2}, \sin\frac{3\pi}{2}, \frac{\sin 4\pi}{2}, \sin\frac{5\pi}{2}$

$1, 0, -1, 0, 1$

c) $a_n = (-1)^{n+1}\left(\frac{2}{n}\right)$

$(-1)^2\left(\frac{2}{1}\right), (-1)^3\left(\frac{2}{2}\right), (-1)^4\left(\frac{2}{3}\right), (-1)^5\left(\frac{2}{4}\right), (-1)^6\left(\frac{2}{5}\right)$

$2, -1, \frac{2}{3}, -\frac{1}{2}, \frac{2}{5}$

2. Write the rule for the sequence.

a) 2, 5, 8, 11, …

$a_n = \{3n - 1\}$

b) $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots$

$a_n = \left\{\frac{n+1}{n+2}\right\}$

3. Simplify the ratio of factorials.

a) $\frac{(n+1)!}{n!} \to \frac{(n+1)n!}{n!} \to n+1$

b) $\frac{(2n-1)!}{(2n+1)!}$ $\frac{(2n-1)!}{(2n+1)(2n)(2n-1)!}$ $\boxed{\to \frac{1}{2n(2n+1)}}$

4. Determine the convergence of the sequence with the given rule. If the sequence converges, find its limit.

a) $a_n = \left\{\frac{5}{n+2}\right\}$

$\lim_{n\to\infty} a_n = 0$ converges to 0

b) $a_n = \left\{\frac{(n+1)!}{n!}\right\}$ $\frac{(n+1)(\cancel{n!})}{\cancel{n!}}$

$\lim_{n\to\infty} a_n \to \infty$ diverges

c) $a_n = \left\{(-1)^n \frac{n}{n+1}\right\}$

$\lim_{n\to\infty} (-1)^n \cdot \lim_{n\to\infty} \frac{n}{n+1} \to (-1)^n \cdot 1$

$\to$ oscillating $\to$ diverges

d) $a_n = \left\{\frac{(-5)^n}{6^n}\right\}$ $\lim_{n\to\infty} \frac{(-1)^n(5)^n}{(6)^n} \to 0$

bottom heavy

converges to 0

e) $a_n = \left\{\frac{10n^2 + 3n + 7}{2n^2 - 6}\right\}$

$\lim_{n\to\infty} a_n = \frac{10}{2} = 5$ converges to 5

f) $a_n = \left\{\frac{\cos(\pi n)}{n^2}\right\} \to \frac{1 \Leftrightarrow -1}{\infty}$

$\lim_{n\to\infty} a_n = 0$ converges to 0

g) $a_n = \left\{\frac{\ln(n^3)}{2n}\right\} \to \frac{\infty}{\infty}$

$\lim_{n\to\infty} \frac{\ln(n^3)}{2n} \xrightarrow{\text{L'H.}} \frac{\frac{1}{n^3}(3n^2)}{2} \to \frac{3n^2}{2n^3} \to \frac{3}{2n} \to 0$

converges to 0

h) $a_n = \left\{2^{\frac{1}{n}}\right\}$ $\frac{1}{n}\ln 2 \to \frac{\ln 2}{n} \to 0$

undo ln

$e^0 = 1$ Converges to 1

5. Find the area of the following polar graphs

a) Inside $r = 3\cos\theta$ outside $r = 2 - \cos\theta$

$3\cos\theta = 2 - \cos\theta$

$4\cos\theta = 2$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}, -\frac{\pi}{3}$

$A = \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left((3\cos\theta)^2 - (2-\cos\theta)^2\right) d\theta$

≈ 5.196

b) Common interior of $r = 3\cos\theta$ and $r = 1 + \cos\theta$

$\frac{1}{2}\int_0^{\pi} (3\cos\theta)^2 d\theta - \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left((3\cos\theta)^2 - (1+\cos\theta)^2\right) d\theta \approx 3.927$

$3\cos\theta = 1 + \cos\theta \quad 2\cos\theta = 1 \quad \cos\theta = \frac{1}{2}$

c) The area outside the circle $r = 3$ and inside one petal of $r = 6\sin(3\theta)$

$3 = 6\sin 3\theta$

$\frac{1}{2} = \sin 3\theta$

$\frac{\pi}{6}, \frac{5\pi}{6} + 2\pi n = 3\theta$

$\theta = \frac{\pi}{18}, \frac{5\pi}{18} + \frac{2\pi n}{3}$

$A = \frac{1}{2}\int_{\frac{\pi}{18}}^{5\pi/18} \left((6\sin 3\theta)^2 - 3^2\right) d\theta \approx$

5.740

6. Find the slope of $r = 2 - 3\sin\theta$ at $\theta = \frac{\pi}{4}$.

$\frac{dr}{d\theta} = -3\cos\theta$ $\quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

$x = r\cos\theta$

$\frac{dx}{d\theta} = r'\cos\theta - r\sin\theta \rightarrow \left(-3\cos\frac{\pi}{4}\right)\left(\cos\frac{\pi}{4}\right) - \left(2 - 3\sin\frac{\pi}{4}\right)\left(\sin\frac{\pi}{4}\right)$

$-\frac{3\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} - \left(2 - \frac{3\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2}) \rightarrow -\frac{3}{2} - \left(\frac{2\sqrt{2}}{2} - \frac{3\cdot 2}{4}\right)$

$-\frac{3}{2} - \sqrt{2} + \frac{3}{2}$

$= -\sqrt{2}$

$y = r\sin\theta \quad \frac{dy}{d\theta} = r'\sin\theta + r\cos\theta$

$-\frac{3}{2} + \sqrt{2} - \frac{3}{2} = -3 + \sqrt{2}$

$\boxed{\frac{dy}{dx} = \frac{-3+\sqrt{2}}{-\sqrt{2}}}$

7. Convert the following points to the indicated form.

a) $(-5,5)$ to polar

$x = r\cos\theta$

$y = r\sin\theta$

(x, y)

$\tan\theta = \frac{y}{x}$

$x^2 + y^2 = r^2$

$\tan\theta = \frac{5}{-5} = -1$

$50 = r^2$

$r = 5\sqrt{2}$

$\theta = \frac{3\pi}{4}$

$\left(5\sqrt{2}, \frac{3\pi}{4}\right)$

b) $\left(-1, \frac{\pi}{3}\right)$ to rectangular

$x = -1\cos\frac{\pi}{3}$

$= -1 \cdot \frac{1}{2} = -\frac{1}{2}$

$y = -1\sin\frac{\pi}{3}$

$= -1 \cdot \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$

$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

8. Write two points in polar form that are equivalent to $\left(-1, \frac{\pi}{3}\right)$, one that has a positive r, and one with a negative r.

$\left(-1, \frac{7\pi}{3}\right) \qquad \left(1, \frac{4\pi}{3}\right)$

BC Calculus Series Convergence/Divergence A Notesheet **Name:** ______________________

A series is the sum of the terms in a sequence. Finite sequences and series have defined first and last terms, whereas infinite sequences and series continue indefinitely. Informally, a series is the result of adding any number of terms from a sequence together: $a_1 + a_2 + a_3 + \cdots$. A series can be written more succinctly by using the summation symbol.

$a_n = a_1, a_2, a_3, \ldots a_n$ $\quad S_n = \sum_{n=1}^{\infty} a_n$

For infinite series, we can look at the sequence of partial sums, that is, looking to see what the sums are doing as we add additional terms. In general, the nth partial sum of a series is denoted S_n. This can be explored on a calculator by adding sequential terms to the aggregate sum.

Example 1 For both $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n^2}$, generates the sequence of partial sums $S_1, S_2, S_3, \ldots, S_n$, for each, then determine if the series converge or diverge. Where else have we seen something like this before?

$S_1 = 1$

$S_2 = 1.5 \rightarrow 1 + \frac{1}{2}$

$S_{100} = 5.187$

$S_{1000} = 7.485$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$S_1 = 1$

$S_2 = 1.25$

$S_{100} = 1.635$

$S_{1000} = 1.644$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges to 1.645

Example 2 Given the series

$$\sum_{n=1}^{\infty} \frac{3}{2^n} = \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16} + \frac{3}{32} + \frac{3}{64} + \frac{3}{128} + \frac{3}{256} + \cdots,$$

Find the first 10 terms of the sequence of partial sums, and list them below, $S_1, S_2, S_3, \ldots, S_{10}$. Based on this sequence of partial sums, do you think the series converges or diverges? To what? (Hint: first rewrite the rule of the sequence so that it looks like an exponential function of n)

$S_{10} = 2.997$ $\quad S_{100} = 3$

$\sum_{n=1}^{\infty} 3 \cdot \frac{1}{2^n} \rightarrow 3 \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$

Example 3 Given the series

$$\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n = \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \frac{81}{16} + \frac{243}{32} + \cdots,$$

Find the first 5 terms of the sequence of partial sums, and list them below, $S_1, S_2, S_3, \ldots, S_5$. Based on this sequence of partial sums, do you think the series converges or diverges? To what?

$S_{10} = 169.995$

$S_{100} = 1.29 \times 10^{18}$

diverges

We are now going to look at several families of infinite series and several tests that will help us determine whether they converge or diverge. For some that converge, we might be able to give the actual sum, or an interval in which we know the sum will be. For others, simply knowing that they converge will have to suffice.

Geometric Series Test (GST)

A geometric series is in the form

$$\sum_{n=0}^{\infty} a \cdot r^n \text{ or } \sum_{n=1}^{\infty} a \cdot r^{n-1}, \quad a \neq 0$$

The geometric series diverges if $|r| \geq 1$.

If $|r| < 1$, the series converges to the sum $S = \frac{a_1}{1-r}$.

Where a_1 is the first term, regardless of where n starts, and r is the common ratio.

Example 4 Using the GST, determine whether each series converges or diverges. If it converges, find the sum.

a) $\sum_{n=1}^{\infty} \frac{3}{2^n}$

$r = \frac{1}{2}$

$S = \frac{3/2}{1 - 1/2} = \frac{3/2}{1/2} = 3$

by GST

b) $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$ $r = \frac{3}{2}$

$\frac{3}{2} > 1$

diverges by GST

c) $\sum_{n=2}^{\infty} 3\left(-\frac{1}{2}\right)^n$ $r = -\frac{1}{2}$

$S = \frac{3/4}{1 - -\frac{1}{2}}$

$= \frac{3}{4} \circ \frac{2}{3}$

$= \frac{1}{2}$ by GST

nth Term Test for Divergence

If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Note: This does not say that if $\lim_{n\to\infty} a_n = 0$, then the series converges. This test can only be used to prove that a series diverges, hence the name. If $\lim_{n\to\infty} a_n \neq 0$, then this test does not tell us anything, is inconclusive, does not work, fails, etc. We must use another test. This test can be a great time-saver. Always perform it first.

Example 5 Use the nth term test to determine whether the following series diverge.

a) $\sum_{n=1}^{\infty} \frac{2n+3}{3n-5} \to a_n$

$\lim_{n\to\infty} a_n = \frac{2}{3} \neq 0$

therefore $\sum_{n=1}^{\infty} \frac{2n+3}{3n-5}$ diverges by nth term test

b) $\sum_{n=1}^{\infty} \frac{n!}{2n! + 1}$ $\lim_{n\to\infty} \frac{n!}{2n!+1} = \frac{1}{2} \neq 0$

same

Integral Test

If f is decreasing, continuous, and positive for $x \geq 1$ and $a_n = f(x)$, then

$$\sum_{n=1}^{\infty} a_n \text{ and } \int_1^{\infty} f(x)dx$$

Either both converge or diverge.

Note 1: This does not mean that the series converges to the value of the definite integral.

Note 2: The function need only be decreasing for all $x > k$ for some $k \geq 1$.

Example 7 Determine whether the following series converge or diverge.

a) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

$\int_1^{\infty} \frac{x}{x^2+1} dx$

$\lim_{b\to\infty} \int_1^b \frac{x}{x^2+1} dx$

$u = x^2+1$

$\frac{du}{dx} = 2x$ $\quad \frac{du}{2} = x\,dx$

$\frac{1}{2}\int \frac{du}{u}$

$\frac{1}{2}\ln|u| \to \frac{1}{2}\ln|x^2+1|$

$\frac{1}{2}\ln|b| - \frac{1}{2}\ln(1+1)$

$\infty - \frac{1}{2}\ln 2$

diverges

diverges by integral test

b) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$\int_1^{\infty} \frac{1}{x^2+1} dx$

$\lim_{b\to\infty} = \tan^{-1} x \Big|_1^b$

$\tan^{-1} b - \tan^{-1} 1$

$\pi/2 - \pi/4 = \pi/4$

converges

So also converges by integral test

P-Series Test

A series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p}$$

Is called a p-series, where p is a positive constant. If $p = 1$, the series is called the harmonic series.

- If $p \leq 1$ the series will diverge.
- If $p > 1$ the series will converge.

Note: If the p-series converges and starts at $n = 1$, we cannot find its sum using $\frac{1}{p-1}$ like we could with p-series integrals.

Example 8 Use the nth term test to determine whether the following series diverge.

a) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

$\frac{1}{n^{1.5}}$ $\quad p = 1.5$

$1.5 > 1$

converges by P-series test

b) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n}$

$\frac{n^{.5}}{n} = \frac{1}{n^{.5}}$ $\quad p = .5$

$.5 < 1$

diverges by p-series test

c) $\sum_{n=1}^{\infty} \frac{3^n - 2}{3^n}$

$\lim_{n\to\infty} \frac{3^n-2}{3^n} = 1 \neq 0$

same explanation

d) $\sum_{n=2}^{\infty} \frac{1}{(1.1)^n}$

$\lim_{n\to\infty} \frac{1}{(1.1)^n} = 0$ inconclusive

$r = \frac{1}{1.1} < 1$ $S = \frac{(1.1)^2}{1-\frac{1}{1.1}} =$

converges by GST

Telescoping Series

A series such as $\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots$ is called a telescoping series because it collapses to one term or just a few terms. If a series collapses to a finite sum, then it converges by the Telescoping Series Test. Write out terms of the series until both the start and ending terms cancel out. Then add the terms that do not cancel out to find the sum of the series.

Example 6 Determine whether the following series converge or diverge. IF they converge, find their sum.

a) $\sum_{n=1}^{\infty} \left(\frac{1}{2n+1} - \frac{1}{2n+3}\right)$

$\frac{1}{2+1} - \frac{1}{2+3}$

$\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{9}\right)+\ldots = \frac{1}{3}$ by telescoping

b) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$\frac{1}{n} + \frac{-1}{n+1}$

$\frac{A}{n} + \frac{B}{n+1} = \frac{1}{n(n+1)}$

$A(n+1) + Bn = 1$ $B = -1$ $A = 1$

$S = \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\ldots$

$= 1$

c) $\sum_{n=1}^{\infty} \frac{1}{n^2+4n+3}$

$\frac{-1}{2(n+3)} + \frac{1}{2(n+1)}$

$\frac{A}{n+3} + \frac{B}{n+1} = \frac{1}{n^2+4n+3}$

$A(n+1) + B(n+3) = 1$ $B = \frac{1}{2}$

$A = -\frac{1}{2}$

$\left(-\frac{1}{8}+\frac{1}{4}\right)+\left(-\frac{1}{10}+\frac{1}{6}\right)+\left(-\frac{1}{12}+\frac{1}{8}\right)+\left(-\frac{1}{14}+\frac{1}{10}\right)$

$= \frac{1}{4}+\frac{1}{6} = 5/12$

AP Calculus Series Convergence Homework A Name: Key

1. Determine the convergence of the sequence with the given rule. If the sequence converges, find its limit.

a) $a_n = \dfrac{4n^3}{3 + 2n + 3n^3}$

$\lim\limits_{n\to\infty} a_n = \frac{4}{3}$ converges to $\frac{4}{3}$

b) $a_n = \dfrac{(3n-1)!}{(3n+2)!}$ $\dfrac{\cancel{(3n-1)!}}{(3n+2)(3n+1)(3n)\cancel{(3n-1)!}}$

$\lim\limits_{n\to\infty} \dfrac{1}{(3n+2)(3n+1)(3n)} = 0$ converges to 0

c) $a_n = \dfrac{4^n}{5 + 6^n}$ EBM $\dfrac{4^n}{6^n}$ bottom heavy

$\lim\limits_{n\to\infty} a_n = 0$

d) $a_n = \dfrac{\ln(3^n)}{n}$ L'H $\dfrac{\frac{1}{3^n}\cdot 3^n \cdot \ln 3}{1}$ $= \ln 3$ converges to $\ln 3$

2. Determine the convergence of the series using the geometric series test. If it converges, find its sum.

a) $\sum\limits_{n=1}^{\infty} \left(\frac{4}{3}\right)^n$ or $\frac{4}{3} > 1$

$\lim\limits_{n\to\infty} \left(\frac{4}{3}\right)^n \neq 0$

Diverges by n^{th} term test

b) $\sum\limits_{n=1}^{\infty} \frac{1}{3^n}$ $r = \frac{1}{3} < 1$ Converges to $\dfrac{1/3}{1 - 1/3} = \dfrac{\frac{1}{3}}{2/3} = 1/2$

c) $\sum\limits_{n=1}^{\infty} \frac{(-7)^n}{6^n} \to \left(\frac{-7}{6}\right)^n$ $\left|\frac{-7}{6}\right| > 1$

diverges by GST

d) $\sum\limits_{n=1}^{\infty} \frac{4}{6^n}$ $4\left(\frac{1}{6}\right)^n$ $\frac{1}{6} < 1$

converges to $\dfrac{4/6}{1 - 1/6} = \dfrac{4/6}{5/6} = 4/5$

3. Find the sum of the telescoping series.

a) $\sum\limits_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right) = \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right)$

$1 + \frac{1}{2} = 3/2$

b) $\sum\limits_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$ $\dfrac{1}{4n-3} - \dfrac{1}{4n+1}$

$\dfrac{A}{4n-3} + \dfrac{B}{4n+1} = \dfrac{4}{(4n-3)(4n+1)}$

$A(4n+1) + B(4n-3) = 4$

$n \to -1/4$ $B = -1$

$n \to 3/4$ $A = 1$

$S_n = \left(\frac{1}{1} - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{13}\right.$ ....

$S_n = 1$

4. Find the area of the following regions

a) Inside $r = 3 + 2\cos\theta$

$$A = \frac{1}{2}\int_0^{2\pi} (3+2\cos\theta)^2 \, d\theta \approx 34.558$$

b) Inside one loop of $r^2 = 10\cos(2\theta)$

$r = \sqrt{10\cos(2\theta)}$

$0 = 10\cos 2\theta$

$0 = \cos 2\theta$

$\pm\frac{\pi}{2} + \pi n = 2\theta$

$\pm\frac{\pi}{4} + \frac{\pi}{2} n = \theta$

$$A = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 10\cos(2\theta)\, d\theta$$

≈ 5

c) Common interior of $r = 2$ and $r = 3 + 2\cos\theta$

$2 = 3 + 2\cos\theta$

$-1 = 2\cos\theta$

$\cos\theta = \frac{1}{2}$ $\quad \theta = \frac{2\pi}{3}, \frac{-2\pi}{3}$

$$A = \frac{1}{2}\int_0^{2\pi} \cancel{(3+2\cos\theta)^2 d\theta} - \frac{1}{2}\int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} \left((3+2\cos\theta)^2 - 2^2\right) d\theta$$

≈ 10.370

5. Convert the following equations to polar form.

a) $x^2 + y^2 = 2y$

$r^2 = 2r\sin\theta$

$r = 2\sin\theta$

b) $xy = 3$

$r\cos\theta \cdot r\sin\theta = 3$

$r^2 = \dfrac{3}{\cos\theta \cdot \sin\theta}$ or $r = \sqrt{\dfrac{3}{\cos\theta\sin\theta}}$

6. Convert the following equations to rectangular form

a) $r \cdot r = 10\sin\theta \cdot r$

$r^2 = 10r\sin\theta$

$x^2 + y^2 = 10y$ or

$x^2 + y^2 - 10y = 0$

$x^2 + (y-5)^2 = 25$

b) $2 = \tan\theta$

$2 = \dfrac{\sin\theta}{\cos\theta}$

$r \cdot 2\cos\theta = \sin\theta \cdot r$

$2x = y$

7. If $r = \theta^3 + 2\cos\theta$ find $\frac{dr}{d\theta}$ at $\theta = \frac{\pi}{6}$.

$$\frac{dr}{d\theta} = 3\theta^2 - 2\sin\theta$$

$$\left.\frac{dr}{d\theta}\right|_{\theta = \frac{\pi}{6}} = 3\left(\frac{\pi}{6}\right)^2 - 2\left(\sin\frac{\pi}{6}\right) = \frac{3\cdot\pi^2}{36} - 2\cdot\frac{1}{2} = \frac{\pi^2}{12} - 1$$

BC Calculus Series Convergence/Divergence B Notesheet **Name:** Key

Direct Comparison Test (DCT)

If $a_n \geq 0$ and $b_n \geq 0$,

If $\sum_{n=1}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$, then $\sum_{n=1}^{\infty} a_n$ converges.

If $\sum_{n=1}^{\infty} a_n$ diverges and $0 \leq a_n \leq b_n$, then $\sum_{n=1}^{\infty} b_n$ diverges.

Note: You must state/show the inequality when stating the conclusion of this test.

Example 1 Determine whether the following series converge or diverge.

a) $\sum_{n=1}^{\infty} \frac{n^3}{n^3+1}$

$0 < n^3 \leq n^3+1$

nth term test

$\sum_{n=1}^{\infty} n^3$ diverges

by DCT

b) $\sum_{n=1}^{\infty} \frac{1}{n^3}$

converges by PST

$p = 3, 3 > 1$

c) $\sum_{n=1}^{\infty} \frac{1}{3^n+2}$

$0 < \frac{1}{3^n+2} \leq \frac{1}{3^n}$

$\sum_{n=1}^{\infty} \frac{1}{3^n} \to \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$

Converges by GST since $r = \frac{1}{3}$ $r < 1$

$\sum_{n=1}^{\infty} \frac{1}{3^n+2}$ converges by DCT

d) $\sum_{n=4}^{\infty} \frac{1}{\sqrt{n}-1}$

$0 < \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{n}-1}$

$\sum_{n=4}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=4}^{\infty} \frac{1}{n^{\frac{1}{2}}}$

$p = \frac{1}{2}$ $\frac{1}{2} < 1$ diverges by PST

$\sum_{n=4}^{\infty} \frac{1}{\sqrt{n}-1}$ diverges by DCT

e) $\sum_{n=1}^{\infty} \frac{|\cos n|}{2^n}$

$0 \leq \frac{|\cos n|}{2^n} \leq \frac{1}{2^n}$

$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ $r = \frac{1}{2}$ $\frac{1}{2} < 1$

converges by GST

$\sum_{n=1}^{\infty} \frac{|\cos n|}{2^n}$ converges by DCT

f) $\sum_{n=2}^{\infty} \frac{1}{n^4-10}$

$0 \leq \frac{1}{n^4} \leq \frac{1}{n^4-10}$

$\sum_{n=2}^{\infty} \frac{1}{n^4}$ PST converges

DCT $\to$ inconclusive

$\lim_{n\to\infty} \frac{\frac{1}{n^4-10}}{\frac{1}{n^4}} \to \frac{n^4}{n^4-10} \to 1$

$0 < 1 < \infty$

LCT

$\sum_{n=2}^{\infty} \frac{1}{n^4-10}$ converges

Limit Comparison Test (LCT)

If $a_n \geq 0$ and $b_n \geq 0$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ or $\lim_{n\to\infty} \frac{b_n}{a_n} = L$, where L is both finite and positive, then the two series

$$\sum_{n=1}^{\infty} a_n \ or \ \sum_{n=1}^{\infty} b_n$$

either both converge or both diverge.

Note: You must show the limit when stating the conclusion of this test.

Example 2 Determine whether the following series converge or diverge.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^2 - 4n + 5}$

$\lim_{n\to\infty} \frac{\frac{1}{3n^2-4n+5}}{\frac{1}{3n^2}} = \frac{1}{3}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ PST converges $2>1$

by LCT original also converges

b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{4n^5 - n^3 + 7}$

$\lim_{n\to\infty} \frac{\frac{n^4}{4n^5-n^3+7}}{\frac{1}{n}} = \frac{1}{4}$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges $p=1$ $1 \leq 1$ PST

$\sum_{n=1}^{\infty} \frac{n^4}{4n^5-n^3+7}$ diverges by LCT

c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3 - 2}$

$\lim_{n\to\infty} \frac{\frac{1}{n^3-2}}{\frac{1}{n^3}} = 1$

$\lim_{n\to\infty} \sum \frac{1}{n^3}$ converges

$p=3$ $3>1$

PST

converges by LCT

d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n - 2}}$

$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{3n-2}}}{\frac{1}{\sqrt{n}}} = \frac{1}{\sqrt{3}}$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ $p=\frac{1}{2}$ $\frac{1}{2}<1$ diverges by PST

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n-2}}$ diverges by LCT

Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series of nonzero terms.

$\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$

The ratio test is inconclusive if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$

good for factorials and exponentials

Example 3 Determine whether the following series converge or diverge.

a) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

$\lim_{n\to\infty} \left| \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} \right| < 1 \;? \rightarrow \frac{2}{n+1} \rightarrow 0 < 1$

converges by ratio test

b) $\sum_{n=1}^{\infty} \frac{n^2(3^n+1)}{2^n}$

$\lim_{n\to\infty} \frac{n^2(3^n+1)}{2^n} \neq 0$

diverges by nth term test

c) $\sum_{n=1}^{\infty} \frac{(n+1)!}{3^n}$

$\lim_{n\to\infty} \left| \frac{\frac{(n+2)!}{3^{n+1}}}{\frac{(n+1)!}{3^n}} \right| \rightarrow \lim_{n\to\infty} \frac{(n+2)}{3} \rightarrow \infty > 1$

diverges ratio test

d) $\sum_{n=1}^{\infty} \frac{3^{n-1}}{n \cdot 2^n}$

$\lim_{n\to\infty} \left| \frac{\frac{3^n}{(n+1)2^{n+1}}}{\frac{3^{n-1}}{n \cdot 2^n}} \right|$

$\rightarrow \lim_{n\to\infty} \frac{3n}{(n+1)2} \rightarrow \frac{3}{2} > 1$

diverges ratio test

Root Test

Let $\sum_{n=1}^{\infty} a_n$ be a series of nonzero terms.

$\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$

The root test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$

good when entire sequence term can be written to nth power

Example 4 Determine whether the following series converge or diverge.

a) $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$ $\quad \sum_{n=1}^{\infty} \left(\frac{e^2}{n}\right)^n$

$\lim_{n\to\infty} \sqrt[n]{\left(\frac{e^2}{n}\right)^n} \rightarrow \frac{e^2}{n} = 0$

$0 < 1$ converges root test

b) $\sum_{n=1}^{\infty} \left(\frac{3n+4}{2n}\right)^n$

$\lim_{n\to\infty} \sqrt[n]{\left(\frac{3n+4}{2n}\right)^n} \rightarrow \frac{3n+4}{2n}$

$\rightarrow \frac{3}{2} > 1$

diverges root test

BC Calculus Series Convergence Homework B Name: Key

1. Use the nth term test to see if the series diverges.

a) $\sum_{n=1}^{\infty} \frac{n}{e^n}$ $\lim_{n\to\infty} \frac{n}{e^n} \to \frac{\infty}{\infty}$

L'H. $\lim_{n\to\infty} \frac{1}{e^n} \to 0$ inconclusive

b) $\sum_{n=1}^{\infty} \frac{n}{\sqrt[3]{n}}$ $\lim_{n\to\infty} \frac{n}{\sqrt[3]{n}} \to \frac{\infty}{\infty}$

L'H. $\lim_{n\to\infty} \frac{1}{\frac{1}{3}n^{-2/3}} \to \sqrt[3]{3n^2} \to \infty \neq 0$

divergent

2. Determine the convergence of the series using the Integral test.

a) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ $\int_1^{\infty} \frac{\ln(n)}{n} dn \to \lim_{b\to\infty} \int_1^b \cdots$

$u = \ln n$

$\frac{du}{dn} = \frac{1}{n}$

$du = \frac{1}{n} dn$

$\int u\,du \to \frac{u^2}{2} + C$

$\left.\frac{(\ln n)^2}{2}\right|_1^b \to \infty - 0$

divergent by integral test

b) $\sum_{n=1}^{\infty} e^{-2n}$ $\int_1^{\infty} e^{-2x} dx$

$\lim_{b\to\infty} \int_1^b e^{-2x} dx$

$\to -\frac{1}{2}e^{-2x}\Big|_1^b \to \left.\frac{-1}{2e^{2x}}\right|_1^b$

$\to 0 + \frac{1}{2e^2}$; Converge by integral test

3. Determine the convergence of the series using the p-series test.

a) $\sum_{n=1}^{\infty} \frac{2}{n^{\frac{6}{7}}}$ $p = \frac{6}{7}$ $\frac{6}{7} \leq 1$

diverges by p-series test

b) $\sum_{n=1}^{\infty} \frac{5n^2}{n^4}$ $\to \frac{5}{n^2}$

$p = 2$ $2 > 1$

Converges by PST

4. Determine the convergence of the series.

a) $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n}$ $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ $p = \frac{2}{3}$ $\frac{2}{3} \leq 1$

diverges by PST

b) $\sum_{n=1}^{\infty} \frac{5(-4)^n}{3^n}$ $\sum_{n=1}^{\infty} 5\left(-\frac{4}{3}\right)^n$

$\left|\left(-\frac{4}{3}\right)\right| \geq 1$ diverges Geometric Series test

c) $\sum_{n=1}^{\infty} \frac{2n^2}{1+n^3}$ $\lim_{n\to\infty} \frac{\frac{2n^2}{1+n^3}}{\frac{1}{\cancel{n}}} \cdot \frac{n}{1} \to 2$

since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges PST $p \leq 1$

$\sum_{n=1}^{\infty} \frac{2n^2}{1+n^3}$ diverges LCT

d) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt[3]{n}}$ $\to \frac{1}{n^{\frac{4}{3}}}$ $\frac{4}{3} > 1$

Converges PST

e) $\sum_{n=1}^{\infty} \frac{n^2}{1+n^2}$ $\lim_{n\to\infty} \frac{n^2}{1+n^2} = 1$

$1 \neq 0$

diverges n^{th} term test

f) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$ $\int_2^{\infty} \frac{1}{x\sqrt{\ln x}}\,dx$

$u = \ln x$ $\int \frac{1}{\sqrt{u}}\,du$

$du = \frac{1}{x}dx$ $\int u^{-\frac{1}{2}}\,du$

$\lim_{b\to\infty}$ ... $2\sqrt{u} + C$

$2\sqrt{\ln x}\Big|_2^b$

$2\sqrt{\ln b} - 2\sqrt{\ln 2}$

$\infty - 2\sqrt{\ln 2}$

diverge integral test

5. Find the sum of the telescoping series.

$$\sum_{n=1}^{\infty} \frac{3}{n^2+7n+10}$$

$(n+5)(n+2)$

$B = 1$

$A = -1$

$\frac{A}{n+5} + \frac{B}{n+2} = \frac{3}{n^2+7n+10}$ $\frac{-1}{n+5} + \frac{1}{n+2}$

$A(n+2) + B(n+5) = 3$

$\left(-\cancel{\frac{1}{6}} + \frac{1}{3}\right) + \left(-\cancel{\frac{1}{7}} + \frac{1}{4}\right) + \left(-\cancel{\frac{1}{8}} + \frac{1}{5}\right) + \left(-\cancel{\frac{1}{9}} + \cancel{\frac{1}{6}}\right) \to \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \boxed{\frac{47}{60}}$

6. Find the area of each region

a) Inside one loop of the graph defined by $r = 5\sin(3\theta)$.

$$A = \frac{1}{2}\int_0^{\pi/3} (5\sin 3\theta)^2\,d\theta \approx 6.545$$

b) Between the loops of the graph defined by $r = 4 + 8\cos\theta$.

$4 + 8\cos\theta = 0$

$\cos\theta = -\frac{1}{2}$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

$$A = \frac{1}{2}\int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} r^2\,d\theta - \frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} r^2\,d\theta \approx 133.404$$

c) Common interior of the graphs $r = 2\sin(2\theta)$ and $r = 1$.

$1 = 2\sin 2\theta$

$\frac{1}{2} = \sin 2\theta$

$\left(\frac{\pi}{6}, \frac{5\pi}{6}\right) + 2\pi n = 2\theta$

$\nearrow \theta = \left(\frac{\pi}{12}, \frac{5\pi}{12}\right) + \pi n$

$$A = 4\left[\frac{1}{2}\int_0^{\pi/2} (2\sin 2\theta)^2\,d\theta - \frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \left((2\sin 2\theta)^2 - 1^2\right)d\theta \approx\right.$$

2.457

7. Find the length of the polar curve $r = \theta^3 + \ln\theta$ over the interval $2 \le \theta \le 5$.

$\frac{dr}{d\theta} = 3\theta^2 + \frac{1}{\theta}$

$$L = \int_2^5 \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta \approx 196.441$$

AP Calculus Series Convergence Homework C Name: Key

1. Determine the convergence of the series using one of the comparison tests.

a) $\sum_{n=1}^{\infty} \frac{1}{4^n+3}$

$\lim_{n\to\infty} \frac{\frac{1}{4^n+3}}{\frac{1}{4^n}} \cdot \frac{4^n}{4^n} \to 1$ so original converges by LCT

$\lim_{n\to\infty} \frac{1}{4^n} \to \left(\frac{1}{4}\right)^n$ $r=\frac{1}{4}$ $|\frac{1}{4}|<1$ converge GST

b) $\sum_{n=2}^{\infty} \frac{\ln n}{n}$

$\int_2^{\infty} \frac{\ln x}{x}\,dx$ $u=\ln x$ $du=\frac{1}{x}dx$

$\int u\,du$

$\lim_{b\to\infty} \ldots \frac{u^2}{2} \to \left.\frac{(\ln x)^2}{2}\right|_2^b$

$\infty - \frac{(\ln 2)^2}{2} \to$ diverges

So by integral test, original diverges

2. Determine the convergence of the series using the Ratio Test

a) $\sum_{n=1}^{\infty} \frac{n!}{6^n}$

$\frac{\left|\frac{(n+1)!}{6^{n+1}}\right|}{\left|\frac{n!}{6^n}\right|} \to \frac{n+1}{6} \to \infty$ $\infty > 1$

diverges by ratio test

b) $\sum_{n=1}^{\infty} \frac{n^2}{(2n+1)!}$

$\frac{\left|\frac{(n+1)^2}{(2n+3)!}\right|}{\left|\frac{n^2}{(2n+1)!}\right|} \to \frac{(n+1)^2}{n^2} \cdot \frac{(2n+1)!}{(2n+3)(2n+2)(2n+1)!}$

$\to 0$ $0<1$

converges ratio test

3. Determine the convergence of the series using the Root Test.

a) $\sum_{n=1}^{\infty} \frac{1}{n^n}$

$\sqrt[n]{\frac{1}{n^n}} \to \frac{1}{n} \to 0$

$\hookrightarrow$ converges by root test

b) $\sum_{n=0}^{\infty} \left(\frac{2n}{n+10}\right)^n$

$\sqrt[n]{\left(\frac{2n}{n+10}\right)^n} \to \frac{2n}{n+10} \to 2$

$2>1$

diverges by root test

4. Determine whether the series converges or diverges. State the test being used.

a) $\sum_{n=1}^{\infty} \frac{8}{n+6}$

$\lim_{n\to\infty} \frac{\frac{8}{n+6}}{\frac{1}{n}} \to 1$ $\frac{1}{n} \to$ PST diverges $p=1$ $1 \le 1$

LCT diverges

b) $\sum_{n=1}^{\infty} \frac{1}{(\ln 2)^n}$

$\left(\frac{1}{\ln 2}\right)^n$ $\frac{1}{\ln 2} > 1$

So GST diverges

c) $\sum_{n=1}^{\infty} \frac{3e^n}{n^5}$

$\frac{\infty}{\infty}$ L'H lots of times

$\frac{3e^n}{\text{constant}} \to \infty$

nth term test diverges

d) $\sum_{n=1}^{\infty} \frac{3}{\sqrt[4]{n}}$

$\frac{3}{n^{\frac{1}{4}}}$ $p=\frac{1}{4}$ $\frac{1}{4} < 1$

PST diverges

e) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ $\lim_{n\to\infty}$ $\cdot \frac{n^{\frac{3}{2}}}{1} \to \frac{n^2}{n^2+1} \to 1$

$\frac{1}{n^{\frac{3}{2}}}$ → converges $p > 1$ PST ($p = 3/2$)

so original converge LCT

f) $\sum_{n=2}^{\infty} \frac{5n^3-3n}{n^2(n+2)(n^2+5)}$ $\lim_{n\to\infty}$ $\cdot \frac{n^2}{1} \to 5$

$\frac{1}{n^2}$ → converges $p=2$ $2>1$ PST

So LCT converges

g) $\sum_{n=1}^{\infty} \frac{4}{n^2}$

$p=2$

PST $2>1$

Converges

f) $\sum_{n=2}^{\infty} \frac{5^n}{4^n}$

$\left(\frac{5}{4}\right)^n$ GST

$r = \frac{5}{4}$ $\frac{5}{4} > 1$

diverges

5. Evaluate each of the following integrals.

a) $\int \left(x^4 + \frac{2}{x} - \frac{1}{\sqrt[3]{x}}\right) dx$ $\cancel{2x}$ $x^{-\frac{1}{3}}$

$\frac{x^5}{5} + 2\ln|x| - \frac{3}{2}x^{\frac{2}{3}} + C$

b) $\int \frac{x^4+2x}{x^2} dx \to x^2 + 2 \cdot \frac{1}{x}$

$\frac{x^3}{3} + 2\ln|x| + C$

LIPET

c) $\int (x \ln x)\, dx$

$u = \ln x$ $\quad dv = x\,dx$

$du = \frac{1}{x}dx$ $\quad v = \frac{x^2}{2}$

$\frac{x^2}{2}\ln x - \int \frac{x^2}{2} \cdot \frac{1}{x}\,dx$

$-\frac{1}{2}\int x\,dx \to \frac{x^2}{2}\ln x - \frac{1}{2}\frac{x^2}{2} + C$

$2 \cdot 2 \to 4$

d) $\int \left(x\sqrt{x^2+2}\right) dx$

$u = x^2 + 2$

$\frac{du}{dx} = 2x$ $\quad \frac{du}{2} = x\,dx$

$\frac{1}{2}\int \sqrt{u}\,du$

$\frac{2}{3} \cdot \frac{1}{2} u^{\frac{3}{2}} + C \to \frac{1}{3}(x^2+2)^{\frac{3}{2}} + C$

6. Evaluate each of the following derivatives.

a) $y^2 = \frac{x-4}{x+2}$

$2y\frac{dy}{dx} = \frac{(x+2)(1) - (x-4)(1)}{(x+2)^2}$

$\to \frac{x+2-x+4}{(x+2)^2} \to \frac{6}{(x+2)^2}$

$\frac{dy}{dx} = \frac{6}{(x+2)^2} \cdot \frac{1}{2} \cdot \frac{1}{y}$

b) $x^2y + 2y = 6x$

$x^2\frac{dy}{dx} + 2xy + 2\frac{dy}{dx} = 6$

$\frac{dy}{dx} = \frac{6-2xy}{(x^2+2)}$

7. Find the particular solution to the differential equation $\frac{dy}{dx} = \frac{x+2}{y^2}$ that goes through the point $(2, 0)$.

$\int y^2\,dy = \int (x+2)\,dx$

$\frac{y^3}{3} = \frac{x^2}{2} + 2x + C$

$0 = 2 + 4 + C$ $\quad C = -6$

$\frac{y^3}{3} = \frac{x^2}{2} + 2x - 6$

$y^3 = \frac{3}{2}x^2 + 6x - 18$

$y = \sqrt[3]{\frac{3}{2}x^2 + 6x - 18}$

BC Calculus Alternating Series Notesheet **Name:** Key

An alternating series is a series whose terms are alternately positive and negative on consecutive terms.
Examples:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\cdots \qquad \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} \qquad -1+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}+\cdots+\cdots \qquad \sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n!}$$

In general, just knowing that $\lim\limits_{n\to\infty} a_n = 0$ tells us very little about the convergence of the series $\sum_{n=1}^{\infty} a_n$. However, it turns out that an alternating series must converge if its terms consistently shrink in size to 0.

Alternating Series Test (AST)

If $a_n > 0$, then the alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ or $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges if both of the following conditions are satisfied:

- $\lim\limits_{n\to\infty} a_n = 0$
- $\{a_n\}$ is a decreasing (or non-increasing) sequence. That is, $a_{n+1} \le a_n$ for all $n > k$ for some $k \in \mathbb{Z}$.

Note: This does not say that if $\lim\limits_{n\to\infty} a_n \neq 0$ the series diverges by AST. The AST can only be used to prove convergence. If $\lim\limits_{n\to\infty} a_n \neq 0$, then the series diverges by the nth term test, not the AST.

Example 1 Determine whether the following series converge or diverge.

a) $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}n}{2n-1}$ diverges nth term

$\lim\limits_{n\to\infty} a_n = \frac{1}{2} \neq 0$

b) $\sum\limits_{n=1}^{\infty}\frac{(-1)^n n}{\ln(2n)}$ $\frac{n}{\ln(2n)} \to \frac{1}{\frac{1}{2n}\cdot 2} \to n$

$n\to\infty > 0$

diverges nth term

c) $\sum\limits_{n=1}^{\infty}\frac{\cos(n\pi)}{n}$ → alternating series

$\lim\limits_{n\to\infty}\frac{1}{n} = 0$ converge AST

d) $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}$ → AST

$\frac{1}{n!} \to 0$ converge AST

e) $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{(n-5)^2+1}$ → alternating

$\frac{1}{n^2} \to 0$ converge AST

f) $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ → alternating

$\lim\limits_{n\to\infty}\frac{1}{n} \to 0$

converge AST

Absolute vs. Conditional Convergence

If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

Such a series is called **absolutely convergent**. Notice that if it converges on its own, the alternator only allows it to converge more rapidly.

$\sum_{n=1}^{\infty} a_n$ is **conditionally convergent** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Example 2 Determine whether the following alternating series converge absolutely, conditionally, or diverges.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ $\frac{1}{\sqrt{n}} \to 0$ alt. converge AST

So conditionally convergent

$\frac{1}{\sqrt{n}} \to \frac{1}{n^{\frac{1}{2}}}$ $\frac{1}{2} \text{ p} < 1$ diverge pST

b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n}$

absolutely convergent

$\left(\frac{1}{3}\right)^n$ $\frac{1}{3} < 1$ converge GST

Alternating Series Remainder

If an alternating series satisfies the conditions of the AST, namely that $\lim_{n\to\infty} a_n = 0$ and $\{a_n\}$ is not increasing, and the series has a sum S, then $|R_n| = |S - S_n| < a_{n+1}$, where S_n is the nth partial sum of the series.

In other words, if an alternating series satisfies the conditions of the AST, you can approximate the sum of the series by using the nth partial sum S_n, and your error will have an absolute value not greater than the first term left off, a_{n+1}. This means $|S_n - R_n| \le S \le |S_n + R_n|$

Example 3 Approximate the sum $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$ by using the first six terms and find the maximum error. Use your result to find the interval in which S must lie.

~~$\frac{1}{16}$~~ $S_6 = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720} + \frac{1}{5040}$

$S_6 = \frac{91}{144}$ $\text{Error} = |S_n - S_6| \le \frac{1}{5040}$ $\frac{91}{144} - \frac{1}{5040} \le S_n \le \frac{91}{144} + \frac{1}{5040}$

actual sum

Example 4 Approximate the sum of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$ with an error less than 0.001. $\frac{1}{1000}$

$\frac{1}{1} - \frac{1}{16} + \frac{1}{81} - \frac{1}{256} + \frac{1}{625} - \frac{1}{1296}$

$S_5 = .948$

BC Calculus Series Test Summary Notesheet **Name:** Key

Here is an acronym to help you remember all of the tests:

P P-series. Is the series of the form $\frac{1}{n^p}$?

A Alternating series. Does the series alternate? If it does, are the terms getting smaller and $\lim_{n\to\infty} a_n = 0$?

R Ratio Test. Does the series contain exponentials and/or factorials?

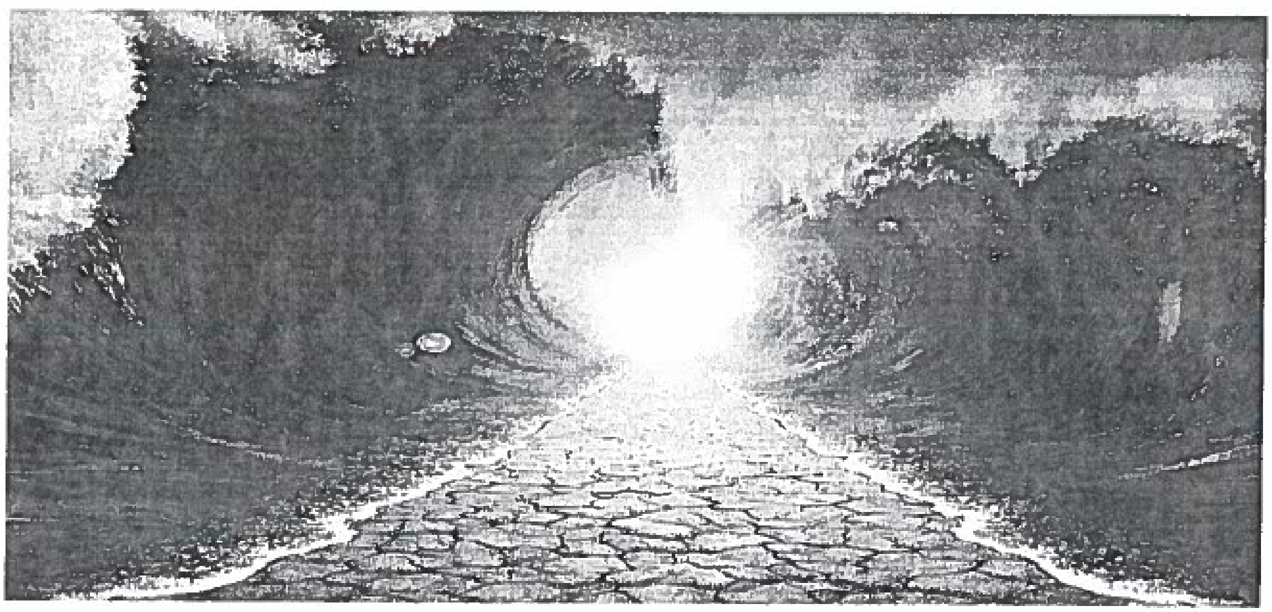
T Telescoping Series. Will all but a couple of the terms in the series cancel out?

I Integral Test. Can you easily integrate the expression that defines the series?

N Nth Term Divergence Test. Is $\lim_{n\to\infty} a_n \neq 0$?

G Geometric Series. Is the series of the form $\sum_{n=0}^{\infty} ar^n$

C Comparison Tests. Is the series almost another kind of series like p-series or geometric? Which would be better to use, Direct or Limit Comparison?



Determine if the following series converge or diverge.

1. $\sum_{n=1}^{\infty} \frac{1+3n^2+n^3}{4n^3-5n+2}$

$\lim_{n\to\infty} a_n = \frac{1}{4}$ $\quad \frac{1}{4} \neq 0$

diverges n^{th} term

2. $\sum_{n=0}^{\infty} \left(\frac{2}{7}\right)^n$

$r = \frac{2}{7}$ $\quad \frac{2}{7} < 1$

Converges

GST

3. $\sum_{n=1}^{\infty} \frac{4}{n^3}$ $\lim_{n\to\infty} a_n = 4$

$\frac{1}{n^3} \to$ PST $3>1$

Converges LCT

4. $\sum_{n=1}^{\infty} \frac{n^2}{5^n}$ $\lim_{n\to\infty}$ $\frac{\infty}{\infty}$ L'H. $\frac{2n}{5^n \cdot \ln 5}$ again $\to \frac{2}{\ln 5 \cdot \ln 5 \cdot 5^n}$

or $\left|\frac{\frac{(n+1)^2}{5^{n+1}}}{\frac{n^2}{5^n}}\right| \to \frac{n^2+2n+1}{n^2} \cdot \frac{1}{5} \to \frac{1}{5}$ $\frac{1}{5}<1$

converges ratio test

5. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^5+5}} < \frac{1}{\sqrt[3]{n^5}} \to$ converges $p=\frac{5}{3}$ $\frac{5}{3}>1$

DCT converges

6. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^4} \to \frac{1}{(\ln n)}$ Nope

$\frac{1}{n}$

$\lim_{b\to\infty} \int_2^{\infty} \frac{1}{x(\ln x)^4}$ $u = \ln x$ $du = \frac{1}{x}\,dx$ $\int \frac{1}{u^4}\,du$ $\frac{u^{-3}}{-3} \to \frac{-1}{3(\ln x)^3}\Big|$ $\to 0 + \frac{1}{3(\ln 2)^3} \to$ Converge Integral Test

7. $\sum_{n=1}^{\infty} \frac{5n^2-6n+3}{n^3-7n+8} \cdot \frac{n}{1} \to 5$ $\lim_{n\to\infty}$

$\frac{1}{n}$ diverges LCT

diverges $p=1$ $1\le 1$

8. $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{\sqrt{n}} \to$ alt.

$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \to 0$ Converges AST

9. $\sum_{n=1}^{\infty} \frac{3^n+4}{2^n} > \frac{3^n}{2^n}$ diverges DCT

$\left(\frac{3}{2}\right)^n$ GST $r>1$ diverges

10. $\sum_{n=1}^{\infty} \frac{8n^3-6n^5}{12n^4-9n^5}$ $\lim_{n\to\infty} \to \frac{-6}{-9} \to \frac{2}{3} \neq 0$

diverges nth term test

11. $\sum_{n=1}^{\infty} \sqrt{\frac{3n+1}{n^5+2}} \cdot \frac{\sqrt{n^4}}{\sqrt{1}} \to \frac{\sqrt{3}}{1} > 0$

$\sqrt{\frac{1}{n^4}} \to \frac{1}{n^2}$ $2>1$ PST converge

So converg LCT

12. $\sum_{n=1}^{\infty} \frac{3^{n-1}}{n2^n} \to$ ~~$\frac{1 \cdot 3^n}{3n2^n}$~~ $\left|\frac{\frac{3^n}{(n+1)2^{n+1}}}{\frac{3^{n-1}}{n2^n}}\right| \to \frac{n3}{2(n+1)} \to \frac{3}{2}>1$

diverges ratio test

13. $\sum_{n=1}^{\infty} \left(\frac{2n}{5n-1}\right)^n$ $\sqrt[n]{\text{all}} = \frac{2n}{5n-1}$

$\frac{2}{5}<1$ converges root test

14. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$ $\sqrt[n]{\frac{1}{(\ln n)^n}} =$

Converge or $\frac{1}{\infty}=0<1$ root test

ok by itself

15. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$ $\sqrt[n]{\text{all}}$

$= 1+\frac{1}{n} = 1$

diverges root $1 \not< 1$

16. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln n}$ alt.

$\lim_{n\to\infty} \frac{1}{\ln n} \to 0$

Converges AST

AP Calculus Alternating Series Homework Name: Key

1. Determine the convergence of the series using the ~~Alternate~~ Alternating Series Test $\to$ alt.

a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{10^n} \to 0$

Converges AST

b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n}} \to 0$

Converges AST

2. Determine whether the given alternating series converges or diverges. If it converges, determine whether it converges conditionally or absolutely.

a) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} \to$ alt.

$\frac{1}{(2n+1)!} \to 0$ converges

$\frac{\frac{1}{(2n+2)!}}{\frac{1}{(2n+1)!}} \to \frac{\cancel{(2n+1)!}}{(2n+2)\cancel{(2n+1)!}}$

$\frac{1}{2n+2} \to 0 < 1$

Converges absolutely (ratio test)

b) $\sum_{n=1}^{\infty} \frac{(-1)^n(1+n)}{n^2}$ $\to$ alt. $\to 0$

$\frac{\frac{1+n}{n^2}}{\frac{1}{n}} \to \frac{n+n^2}{n^2} \to 1$

$\frac{1+n}{n^2} > \frac{1}{n^2} \hookrightarrow$ converges

$\frac{1}{n}$ diverges PST

converges conditionally

3. Approximate the sum of the alternating series using its first five terms. Find the maximum amount of error for this approximation.

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36}$

Error $\leq \frac{1}{36}$

$S \approx S_5 = .838\overline{611} \to \frac{3019}{3600}$

4. Determine whether the series converges or diverges. State the test being used.

a) $\sum_{n=1}^{\infty} \frac{2^n}{n}$

$\lim_{n\to\infty}$ L'H. $\frac{\ln 2 \cdot 2^n}{1} \to \infty \neq 0$

diverges n^{th} term test

b) $\sum_{n=1}^{\infty} 7^{-n}$

$\frac{1}{7^n} \to \left(\frac{1}{7}\right)^n$ $r < \frac{1}{7}$

converges GST

c) $\sum_{n=2}^{\infty} \frac{\cos(n\pi)}{n\ln(n)}$ $\to$ alt.

$\frac{1}{n\ln n} \to 0$

Converges AST

d) $\sum_{n=1}^{\infty} n^{-7.5}$

$\frac{1}{n^{7.5}}$

$p = 7.5$

$p > 1$

converges PST

$\lim_{n\to\infty}$

e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^4+12n} \to \frac{1}{3}$

$\frac{1}{n^4} \to$ converges LCT

$p=4$ $4>1$ PST

f) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n^3(n+2)}$ $\lim_{n\to\infty} \to 1$ diverges LCT

$\frac{1}{n} \to p=1$ diverges PST

g) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3-5}$ $\lim_{n\to\infty} \to \underline{1}$

$\frac{1}{n^3} \to p=3$ $3>1$ converge PST

Converge LCT

h) $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{(n-1)!}$ $\dfrac{\frac{4^{n+1}}{n!}}{\frac{4^n}{(n-1)!}}$ $\frac{4^{n+1}}{n!}\cdot\frac{(n-1)!}{4^n} \to \frac{4}{n-1} \to 0$

$0<1$ converges ratio test

i) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{\sqrt[3]{n}}$ $\to 5$ diverges LCT

$\frac{1}{\sqrt[3]{n}} \to$ diverges $P=\frac{1}{3}$ $\frac{1}{3}\leq 1$ PST

j) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{5^n+1}$ $\cancel{\frac{2^{n+1}}{5^{n+1}+1}}$ $\frac{2^n}{5^n} > \frac{2^n}{5^n+1}$ $\sqrt[n]{\left(\frac{2}{5}\right)^n}$

$\cancel{\frac{2^n}{5^n+1}}$ $\frac{2^n}{5^n+1}$

DCT (bigger converges

$\frac{2}{5}<1$ converges root test

5. Evaluate each of the following integrals.

a) $\displaystyle\int \frac{1}{x^2+5x+6}dx$ $\displaystyle\int\left(\frac{1}{x+2}-\frac{1}{x+3}\right)dx$

$(x+2)(x+3)$

$\frac{A}{x+2}+\frac{B}{x+3}=\frac{1}{x^2+5x+6}$ $=\ln|x+2|-\ln|x+3|+C$

$A(x+3)+B(x+2)=1$

$B=-1$

$A=1$

b) $\displaystyle\int \frac{\sin^{-1}x}{\sqrt{1-x^2}}dx$ $u=\sin^{-1}x$

$du=\frac{1}{\sqrt{1-x^2}}dx$

$\int u\,du \to \frac{u^2}{2}+C$ $\frac{(\sin^{-1}x)^2}{2}+C$

6. Evaluate each of the following limits.

a) $\displaystyle\lim_{x\to 4} \frac{x-4}{x^2-5x+4}$ L'H. $\frac{1}{2x-5}$

$=\frac{1}{3}$

b) $\displaystyle\lim_{x\to 0} \frac{\sin(x^3)}{x}$ L'H.

$\frac{3x^2\cos x^3}{1}=0$

AP Calculus Sequence and Series Review Name: ______________________

Determine the convergence of the sequence. If the sequence converges, find the value it converges to.

1. $a_n = \left\{\frac{2n}{n^2+1}\right\}$

2. $a_n = \left\{\frac{\ln(n)}{\ln(2n)}\right\}$

3. $a_n = \left\{\frac{(n+2)!}{n!}\right\}$

4. $a_n = \left\{\frac{(3)^{n+2}}{5^n}\right\}$

Find the sum of the series

5. $\sum_{n=1}^{\infty} \frac{1+2^n}{3^n}$

6. $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$

Determine the convergence of the series

7. $\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+n}$

8. $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$

9. $\sum_{k=1}^{\infty} k^{-2.4}$

10. $\sum_{k=1}^{\infty} \frac{2^k}{4^k+k}$

11. $\sum_{n=1}^{\infty} \frac{n}{e^n}$

12. $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{2^{3n}}$

13. $\sum_{n=2}^{\infty} \frac{2}{n(lnn)^3}$

14. $\sum_{k=1}^{\infty} \frac{(-1)^k lnk}{\sqrt{k}}$

15. $\sum_{n=1}^{\infty} \frac{(-2)^{2n}}{n^n}$

16. $\sum_{n=1}^{\infty} \frac{2^n}{(2n+1)!}$

Approximate the sum of the alternating series correct to four decimal places

17. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{10^n}$

Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

18. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$

19. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+2}$

20. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^4}{3n^4}$