AB Calculus Area between Curves Notesheet **Name:** ___________________________________

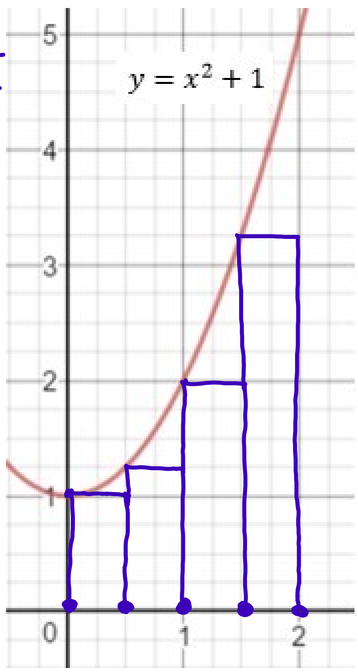
Recall the area under a curve can be approximated through the use of Riemann sums: We can break the area into rectangles, find the area of the rectangles, and add them together. Use this idea to complete the following example and hopefully discover the main idea for today.

Example 1 Find the area of the region between the graphs of $y = x^2 + 1$ and $y = x$ over the interval $[0, 2]$.

a) On the axes provided, show the LRAM approximation of the area under the curve of $y = x^2 + 1$ over the interval $[0, 2]$ using 4 rectangles.

$\frac{2-0}{4} = \frac{1}{2}$

x	$y=x^2+1$
0	1
1/2	5/4
1	2
3/2	13/4
2	5

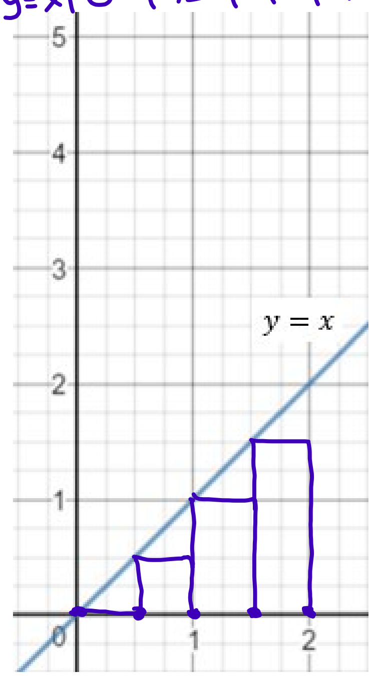


Write an expression to find the exact area under the curve $y = x^2 + 1$ over the interval $[0, 2]$.

$\int_0^2 (x^2+1)\,dx$

b) On the axes provided below, show the LRAM approximation of the area under the curve of $y = x$ over the interval $[0, 2]$ using 4 rectangles.

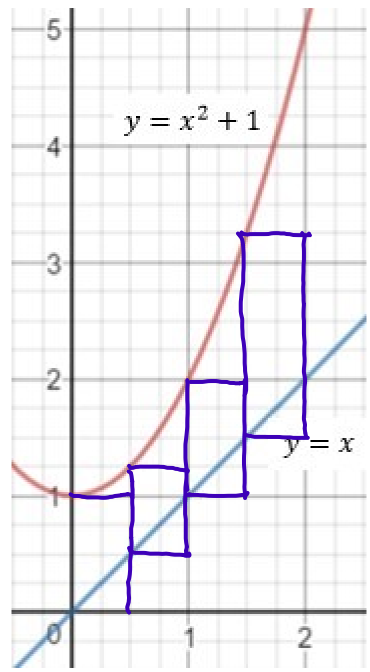
x	0	1/2	1	3/2
$y=x$	0	1/2	1	3/2



Write an expression to find the exact area under the curve $y = x$ over the interval $[0, 2]$?

$\int_0^2 x\,dx$

c) How could you approximate the area between the graphs of $y = x^2 + 1$ and $y = x$ over the interval $[0, 2]$? Draw it on the axes below and shade the approximation.



Write an expression to find the exact area between the graphs of $y = x^2$ and $y = x$ over the interval $[0, 2]$?

$\int_0^2 [(x^2+1) - (x)]\,dx$

Area of a Region Between Two Curves

If f and g are continuous over the interval $[a, b]$ and $g(x) \le f(x)$ for all x in $[a, b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\,dx$$

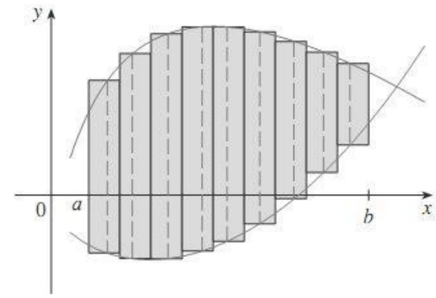
"To find the area between two curves, integrate top minus bottom or, if in terms of y, right minus left."

Another Way of Thinking About It

Area is always positive. Up to now, we've only considered area between a curve and the *x*-axis. We first learned to approximate areas by using rectangular approximations. You can think of each of those rectangles as being a slice of the region. Adding up the areas of all of the slices gives you an approximation of the area of the region.

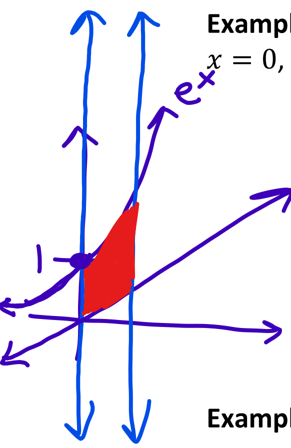
The figure to the right demonstrates what the slices might look like between two curves f and g on an interval $[a, b]$.

If we can find a way to represent the height of each of those **representative rectangular** slices as a positive expression, we can integrate over the interval to find the exact area between the curves.



Steps to find the area between two curves

1. Draw a picture and shade the desired region.
2. Draw an arbitrary rectangular strip, representing one rectangle in a Riemann Sum approximation.
3. Find an expression for the area of that strip (slice)
4. Integrate the area expression for the single slice over the interval to get the area between the curves.



Example 2 Find the area of the region bounded by the functions $\boxed{f(x) = e^x}$ $\boxed{g(x) = x}$, and the vertical lines $x = 0$, and $x = 1$.

Area $= \int_0^1 (e^x - x)\,dx$

$= e^x - \frac{1}{2}x^2 \Big|_0^1 = \left[e^1 - \frac{1}{2}(1)^2\right] - \left[e^0 - \frac{1}{2}(0)^2\right]$

$= e - \frac{1}{2} - 1 + 0$

$\boxed{= e - \frac{3}{2}}$

Example 3 Find the area of the region enclosed by the functions $y = x^2$ and $y = 2x - x^2$.

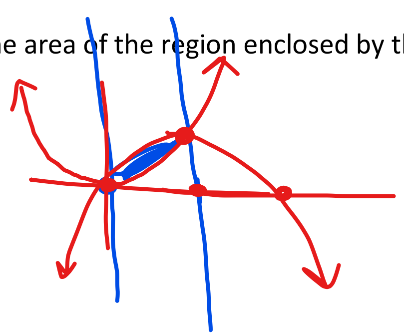
$y = x(2 - x)$

$x^2 = 2x - x^2$

$2x^2 - 2x = 0$

$2x(x - 1) = 0$

$x = 0$ and $x = 1$



$\int_0^1 (2x - x^2 - x^2)\,dx$

$= x^2 - \frac{1}{3}x^3 - \frac{1}{3}x^3 \Big|_0^1 = 1 - \frac{1}{3} - \frac{1}{3}$

$= 1/3$

Example 4 Using your calculator, find the area of the region bounded by $f(x) = \frac{x}{\sqrt{x^2+1}}$ and $g(x) = x^4 - x$.

① Graph (to find the intersection)

② (store the x-value of intersections as A (and B if there are 2)

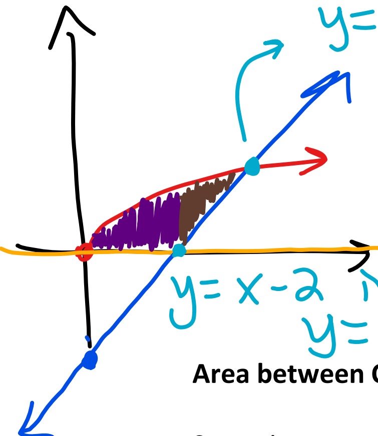
③ $\int_A^B$ (Top Function - Bottom function) dx

$A \approx 1.18..$ (always use a stored value)

Ans: $\int_0^A (f(x) - g(x))\,dx \approx .785$

Sometimes, the top and bottom functions of a region change, requiring us to split up the region.

Example 5 Find the area of the region in the first quadrant bounded by $y = \sqrt{x}$, $y = 0$, and $y = x - 2$.



$y = x-2$ intersects $y = \sqrt{x}$

$x - 2 = \sqrt{x}$ at $x = 4$

$y = x-2$ intersects $y = 0$ at $x = 2$

$$\int_0^2 (\sqrt{x} - 0)\,dx + \int_2^4 \sqrt{x} - (x-2)\,dx$$

$$= \frac{2}{3}x^{\frac{3}{2}}\Big|_0^2 + \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2 + 2x\right]_2^4$$

$$= \frac{2}{3}(2)^{\frac{3}{2}} + \frac{2}{3}(4)^{\frac{3}{2}} - \frac{1}{2}(4)^2 + 2(4) - \frac{2}{3}(2)^{\frac{3}{2}} + \frac{1}{2}(2)^2 - 2(2)$$

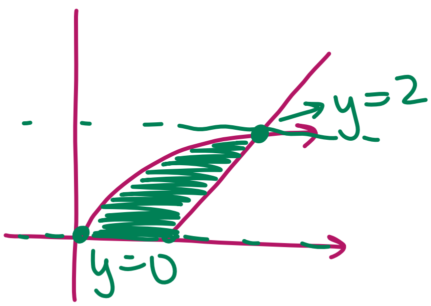
$\frac{2}{3}(8) + 2 - 4 = \frac{16}{3} - \frac{6}{3} = \frac{10}{3}$

Area between Curves of Functions in Terms of y

Sometimes, we will be given equations that are in terms of y or it will be easier to calculate the area if the functions are in terms of y. In either case, the steps are similar, except that we use horizontal rectangular slices instead of vertical, and integrate from bottom to top instead of left to right.

$$\int_{\text{Bottom } y}^{\text{Top } y} (\text{Right} - \text{Left})\,dy$$

Example 6 Find the area of the region in the first quadrant bounded by the graphs of $x = y^2$, $x = y + 2$, and $y = 0$.



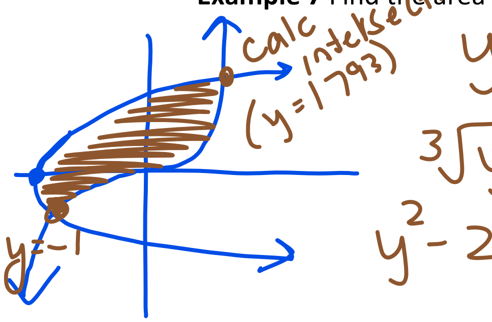
$$\int_0^2 (y + 2 - y^2)\,dy$$

$$= \frac{y^2}{2} + 2y - \frac{y^3}{3}\Big]_0^2 = \frac{2^2}{2} + 2(2) - \frac{2^3}{3} = 2 + 4 - \frac{8}{3}$$

$$= \frac{18}{3} - \frac{8}{3}$$

$$= \frac{10}{3}$$

Example 7 Find the area of the region enclosed by the graphs of $y = x^3$ and $x = y^2 - 2$.



$y = x^3$

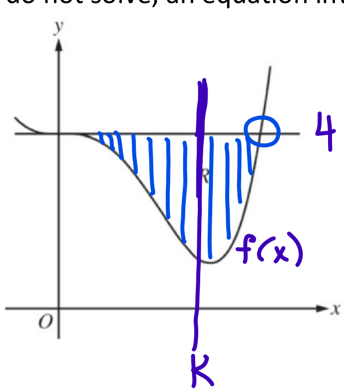
$\sqrt[3]{y} = x$

$y^2 - 2 = x$

$$\int_{-1}^{1.793} \left(\sqrt[3]{y} - (y^2 - 2)\right)dy$$

≈ 4.215 (from calculator)

Example 8 Let R be the region enclosed by the graph of $f(x) = x^4 - 2.3x^3 + 4$ and the horizontal line $y = 4$, as shown in the figure below. The vertical line $x = k$ divides R into two regions with equal areas. Write, but do not solve, an equation involving integral expressions whose solution gives the value of k.



$$\int_0^K (4 - f(x))\,dx = \int_K^{*} (4 - f(x))\,dx$$

* intersection (2.3)

* to find this, $y_1 = f(x)$
$y_2 = 4$
Graph
calc Intersection
(pull the x-value)