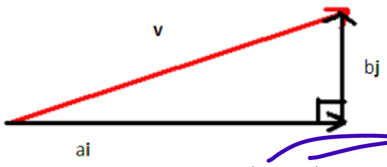
BC Calculus Vector-Valued Functions **Name:** ______________________

Linear Combination Form

Any vector $\mathbf{v} = \langle a, b\rangle$ can be written as a linear combination of the two **standard unit vectors.**

$$\mathbf{i} = \langle 1,0\rangle \qquad \mathbf{j} = \langle 0,1\rangle$$



The vector **v** is a **linear combination** of the vectors **i** and **j**. The scalar a is the **horizontal component** of **v** and the scalar b is the **vertical component** of **v.**

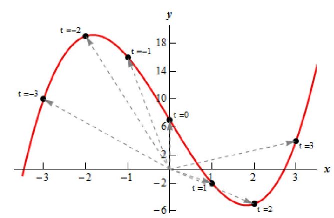
Example 1 Let $P = (-1,5)$ and $Q = (3,2)$. Write the vector $\overrightarrow{PQ}$ as a linear combination of **i** and **j**.

$Q - P \rightarrow (3 - -1, \; 2 - 5)$

$\langle 4, -3 \rangle \leftrightarrow 4i - 3j$

Vector-Valued Functions

A particle moves through the plane on a time interval I with the coordinates of the particle as functions defined by I: $x = f(t)$ $y = g(t)$
The points $(x, y) = (f(t), g(t)), t \in I$ make up the curve in the plane that is the particle's path.

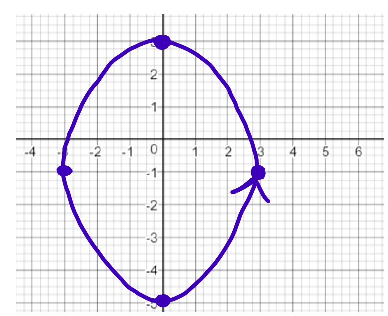


The vector $\mathbf{r}(t) = \overrightarrow{OP} = \langle f(t), g(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j}$ from the origin to the particle's position at time t is the **position vector.** The functions f and g are the **component functions (components)** of the position vector. The particle's path is the curve traced by r during the time interval I.

$r(t)$

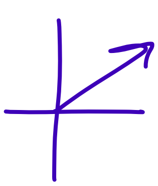
Example 2 Graph the vector function
$r(t) = (3\cos t)\mathbf{i} + (4\sin t - 1)j$ where $t > 0$.

t	0	$\pi/2$	π	$3\pi/2$	2π
$3\cos t$	3	0	-3	0	3
$4\sin t - 1$	-1	3	-1	-5	-1



Limit of a vector function

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$. If $\lim_{t\to c} f(t) = L_1$ and $\lim_{t\to c} g(t) = L_2$, then the limit of $r(t)$ as t approaches c is
$\lim_{t\to c} \mathbf{r}(t) = \boldsymbol{L} = L_1\mathbf{i} + L_2\mathbf{j}.$



Example 3 Let $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$. Find $\lim_{t\to\pi/4} \mathbf{r}(t)$.

$\lim_{t\to\frac{\pi}{4}} \cos t = \frac{\sqrt{2}}{2}$ $\qquad \lim_{t\to\frac{\pi}{4}} \sin t = \frac{\sqrt{2}}{2}$ $\qquad \lim_{t\to\frac{\pi}{4}} r(t) = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$

Example 4 Let $\mathbf{r}(t) = (t^2)\mathbf{i} + \ln(t-3)\,\mathbf{j}$. Find $\lim_{t\to 0} \mathbf{r}(t)$.

$\lim_{t\to 0} t^2 = 0$ $\qquad \lim_{t\to 0} \ln(t-3)$ DNE $\qquad \lim_{t\to 0} r(t)$

DNE

Derivative of a vector function

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ with components made up of the differentiable functions $f(t)$ and $g(t)$. Then, the derivative of $\mathbf{r}(t)$ is

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j}$$

7b) $\int v(t)\,dt = \int (t^2+3)\,dt\, i + \int (-5t-2)\,dt\, j$

$r(t) = \left(\frac{t^3}{3}+3t+c\right)i + \left(-\frac{5}{2}t^2-2t+c\right)j$

$r(0) = (0+0+c)i + (0-0+c)j = 1i + 1j$

$c=1$ $c=1$

$r(t) = \left(\frac{t^3}{3}+3t+1\right)i + \left(-\frac{5}{2}t^2-2t+1\right)j$

Example 5 Let $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$. Find $\mathbf{r}'(t)$.

$r'(t) = (-\sin t)i + (\cos t)j$

Integral of a vector function

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ with components made up of the integrable functions $f(t)$ and $g(t)$. Then, the definite integral of $\mathbf{r}(t)$ from a to b is

$$\int_a^b \mathbf{r}(t)dt = \left(\int_a^b f(t)dt\right)\mathbf{i} + \left(\int_a^b g(t)dt\right)\mathbf{j}$$

Example 6 Evaluate $\int_0^{\pi}((\cos t)\mathbf{i} + 2t\mathbf{j})dt$.

$\int_0^{\pi} (\cos t)\,dt\, i + \int_0^{\pi} 2t\,dt\, j \rightarrow \sin t\Big|_0^{\pi} i + t^2\Big|_0^{\pi} j$

$(\sin\pi - \sin 0)i = 0i$

$(\pi^2 - 0^2)j = \pi^2 j$

$0i + \pi^2 j$ or $\pi^2 j$

Example 7 Find an equation for $\mathbf{r}(t)$ and $\mathbf{v}(t)$ if $\boxed{\mathbf{a}(t) = 2t\mathbf{i} - 5\mathbf{j}}$ and $\boxed{\mathbf{v}(0) = 3\mathbf{i} - 2\mathbf{j}}$ and $\mathbf{r}(0) = \mathbf{i} + \mathbf{j}$.

$\int a(t)\,dt = \int 2t\,dt\, i + \int -5\,dt\, j$

$v(t) = (t^2+c)i + (-5t+c)j$

$v(0) = (0^2+c)i + (-5(0)+c)j = 3i - 2j$

$c=3$ $c=-2$

$v(t) = (t^2+3)i + (-5t-2)j$

$(x, y, \frac{dy}{dx})$ **Example 8** The vector $\mathbf{r}(t) = (2t^3 - 3t^2)\mathbf{i} + (t^3 - 12t)\mathbf{j}$ gives the position of a particle at time t.

a) Write an equation for the line tangent to the path of the particle at the point where $t = -1$.

$r(-1) = (2(-1)^3 - 3(-1)^2)i + ((-1)^3 - 12(-1))j$

$= (-2-3)i + (-1+12)j$

$r(-1) = (-5)i + (11)j$

$r'(t) = (6t^2 - 6t)i + (3t^2 - 12)j$

$r'(-1) = (6(-1)^2 - 6(-1))i + (3(-1)^2 - 12)j$

$r'(-1) = 12i - 9j$

$\frac{dy}{dx} = \frac{-9}{12} = \frac{-3}{4}$

$y - 11 = \frac{-3}{4}(x+5)$

b) Find the coordinates of each point where the horizontal component of the velocity is zero.

$6t^2 - 6t = 0$

$6t(t-1) = 0$ $t=0$ $t=1$

$r(0) = 0i - 0j$ $(0,0)$

$r(1) = -1i - 11j$ $(-1,-11)$

c) Find the acceleration vector at time $t = 2$.

$r''(t) = (12t - 6)i + (6t)j$

$r''(2) = (12\cdot 2 - 6)i + (6\cdot 2)j = 18i + 12j$ or $\langle 18, 12\rangle$