BC Calculus Alternating Series Notesheet **Name:** ______________________________

An alternating series is a series whose terms are alternately positive and negative on consecutive terms. Examples:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\cdots \qquad \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} \qquad -1+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}+\cdots+\cdots \qquad \sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n!}$$

In general, just knowing that $\lim_{n\to\infty} a_n = 0$ tells us very little about the convergence of the series $\sum_{n=1}^{\infty} a_n$. However, it turns out that an alternating series must converge if its terms consistently shrink in size to 0.

Alternating Series Test (AST)

If $a_n > 0$, then the alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ or $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ converges if both of the following conditions are satisfied:

- $\lim_{n\to\infty} a_n = 0$
- $\{a_n\}$ is a decreasing (or non-increasing) sequence. That is, $a_{n+1} \le a_n$ for all $n > k$ for some $k \in \mathbb{Z}$.

Note: This does not say that if $\lim_{n\to\infty} a_n \neq 0$ the series diverges by AST. The AST can only be used to prove convergence. If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges by the nth term test, not the AST.

Example 1 Determine whether the following series converge or diverge.

a) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{2n-1}$ diverges by nth term

$\lim_{n\to\infty}\frac{n}{2n-1} = \frac{1}{2} \neq 0$

b) $\sum_{n=1}^{\infty}\frac{(-1)^{n}n}{\ln(2n)}$ diverge by nth term

$\lim_{n\to\infty}\frac{n}{\ln(2n)} \xrightarrow{L'H} \frac{1}{\frac{1}{2n}(2)} \to n \to \infty$

c) $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n}$ converges by AST

$\lim_{n\to\infty}\frac{1}{n} \to 0$ since $\cos(n\pi)$ alternates between -1 and 1

d) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}$ converges by AST

$\lim_{n\to\infty}\frac{1}{n!} = 0$

e) $\sum_{n=1}^{\infty}\frac{(-1)^{n}}{(n-5)^2+1}$ AST converges

$\lim_{n\to\infty}\frac{1}{(n-5)^2+1} = 0$

f) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges by AST

$\lim_{n\to\infty}\frac{1}{n} \to 0$

Absolute vs. Conditional Convergence

If the series $\sum_{n=1}^{\infty}|a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

Such a series is called **absolutely convergent**. Notice that if it converges on its own, the alternator only allows it to converge more rapidly.

$\sum_{n=1}^{\infty} a_n$ is **conditionally convergent** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty}|a_n|$ diverges.

Example 2 Determine whether the following alternating series converge absolutely, conditionally, or diverges.

a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ $\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$ conditionally converging by AST

as its own series, this is a p-series w a $p=\frac{1}{2}\to$ diverges

b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{3^n}$ $\displaystyle\lim_{n\to\infty}\frac{1}{3^n}=0$ AST converges absolutely

* $\frac{1}{3^n}=\left(\frac{1}{3}\right)^n$

G series w/ an $r=\frac{1}{3}\to$ converges

Alternating Series Remainder

If an alternating series satisfies the conditions of the AST, namely that $\lim_{n\to\infty} a_n = 0$ and $\{a_n\}$ is not increasing, and the series has a sum S, then $|R_n| = |S - S_n| < a_{n+1}$, where S_n is the nth partial sum of the series.

In other words, if an alternating series satisfies the conditions of the AST, you can approximate the sum of the series by using the nth partial sum S_n, and your error will have an absolute value not greater than the first term left off, a_{n+1}. This means $|S_n - R_n| \le S \le |S_n + R_n|$

$a_1=1$ $a_2=\frac{-1}{2}$ $a_3=\frac{1}{6}$ $a_4=\frac{-1}{24}$ $a_5=\frac{1}{120}$ $a_6=\frac{-1}{720}$

Example 3 Approximate the sum $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}$ by using the (a) first six terms and (b) find the maximum error. Use your result to find (c) the interval in which S must lie.

$S_6 = 1-\frac{1}{2}+\frac{1}{6}-\frac{1}{24}+\frac{1}{120}-\frac{1}{720}=\frac{91}{144}$

(a) $S\approx\frac{91}{144}$

(b) $\left|S-\frac{91}{144}\right|<\left|\frac{1}{5040}\right|$

(c) $\frac{91}{144}-\frac{1}{5040}\le S\le\frac{91}{144}+\frac{1}{5040}$

Example 4 Approximate the sum of $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}$ with an error less than 0.001. $\to \frac{1}{1000}$

$a_1=1$ $a_4=\frac{-1}{256}$

$a_2=\frac{-1}{16}$ $a_5=\frac{1}{625}$

$a_3=\frac{1}{81}$ $a_6=\frac{-1}{1296}$ ✓

$S\approx S_5 = 1-\frac{1}{16}+\frac{1}{81}-\frac{1}{256}+\frac{1}{625}$

$=.9475$