BC Calculus Alternating Series Notesheet **Name:** ___________________________

An alternating series is a series whose terms are alternately positive and negative on consecutive terms.
Examples:

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\cdots \qquad \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} \qquad -1+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}+\cdots+\cdots \qquad \sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n!}$$

In general, just knowing that $\lim_{n\to\infty} a_n = 0$ tells us very little about the convergence of the series $\sum_{n=1}^{\infty} a_n$. However, it turns out that an alternating series must converge if its terms consistently shrink in size to 0.

Alternating Series Test (AST)

If $a_n > 0$, then the alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ or $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ converges if both of the following conditions are satisfied:

- $\lim_{n\to\infty} a_n = 0$
- $\{a_n\}$ is a decreasing (or non-increasing) sequence. That is, $a_{n+1} \le a_n$ for all $n > k$ for some $k \in \mathbb{Z}$.

Note: This does not say that if $\lim_{n\to\infty} a_n \neq 0$ the series diverges by AST. The AST can only be used to prove convergence. If $\lim_{n\to\infty} a_n \neq 0$, then the series diverges by the nth term test, not the AST.

Example 1 Determine whether the following series converge or diverge.

a) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{2n-1}$

$\lim_{n\to\infty}\frac{n}{2n-1}=\frac{1}{2}\neq 0$

diverges by nth term

b) $\sum_{n=1}^{\infty}\frac{(-1)^{n}n}{\ln(2n)}$

$\lim_{n\to\infty}\frac{n}{\ln(2n)} \overset{L'H}{\to} \lim_{n\to\infty}\frac{1}{\frac{1}{2n}(2)}$

$\lim_{n\to\infty} n \to \infty$

nth term says diverges

c) $\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n}$

$a_1 = -1$
$a_2 = \frac{1}{2}$
$a_3 = -\frac{1}{3}$

$\lim_{n\to\infty}\frac{1}{n}\to 0$

and the terms alternate in sign $\to$ AST says converge

d) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n!}$

$\lim_{n\to\infty}\frac{1}{n!}\to 0$

what does $(-1)^{n-1}$ do? $\to$ alternate signs

AST $\to$ converge

e) $\sum_{n=1}^{\infty}\frac{(-1)^n}{(n-5)^2+1}$

$\lim_{n\to\infty}\frac{1}{(n-5)^2+1}\to 0$

$a_1 = \frac{1}{17}$ $a_5 = 1$
$a_2 = \frac{1}{10}$ $a_6 = \frac{1}{2}$
$a_3 = \frac{1}{5}$ $a_7 = \frac{1}{5}$
$a_4 = \frac{1}{2}$ $a_8 = \frac{1}{10}$

and $(-1)^n$ alternates the signs

AST $\to$ converges

f) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$

$\lim_{n\to\infty}\frac{1}{n}\to 0$

and $(-1)^{n+1}$ alternates

AST $\to$ converges

Absolute vs. Conditional Convergence

If the series $\sum_{n=1}^{\infty}|a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.

Such a series is called **absolutely convergent**. Notice that if it converges on its own, the alternator only allows it to converge more rapidly.

$\sum_{n=1}^{\infty} a_n$ is **conditionally convergent** if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty}|a_n|$ diverges.

Example 2 Determine whether the following alternating series converge absolutely, conditionally, or diverges.

a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ $\lim_{n\to\infty} \frac{1}{\sqrt{n}} \to 0$ and Alternates from $(-1)^n$ converges conditionally

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$? p-series $\to p = \frac{1}{2} \leq 1 \to$ diverges w/no alternate signs $\to$ therefore

b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n}$ $\lim_{n\to\infty} \frac{1}{3^n} \to 0$

$\sum_{n=1}^{\infty} \frac{1}{3^n} \to$ Geometric Series $r = \frac{1}{3}$ $|r| < 1$ so converges $\to$ therefore convergence is absolute

> **Alternating Series Remainder**
>
> If an alternating series satisfies the conditions of the AST, namely that $\lim_{n\to\infty} a_n = 0$ and $\{a_n\}$ is not increasing, and the series has a sum S, then $|R_n| = |S - S_n| < a_{n+1}$, where S_n is the nth partial sum of the series.
>
> In other words, if an alternating series satisfies the conditions of the AST, you can approximate the sum of the series by using the nth partial sum S_n, and your error will have an absolute value not greater than the first term left off, a_{n+1}. This means $|S_n - R_n| \leq S \leq |S_n + R_n|$

Example 3 Approximate the sum $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$ by using the first six terms and find the maximum error. Use your result to find the interval in which S must lie.

$a_1 = 1$ $a_4 = -\frac{1}{24}$

$a_2 = -\frac{1}{2}$ $a_5 = \frac{1}{120}$

$a_3 = \frac{1}{6}$ $a_6 = \frac{1}{-720}$

$\boxed{S_6 = \frac{91}{144}}$ (6th partial sum)

Error $|S - S_6| \leq \frac{1}{5040}$

$a_7 = \frac{1}{5040}$

$\boxed{\frac{91}{144} - \frac{1}{5040} \leq S \leq \frac{91}{144} + \frac{1}{5040}}$ actual sum

Example 4 Approximate the sum of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$ with an error less than 0.001. $\frac{1}{1000}$

$a_1 = 1$ $a_4 = \frac{-1}{256}$

$a_2 = \frac{-1}{16}$ $a_5 = \frac{1}{625}$

$a_3 = \frac{1}{81}$ $a_6 = \frac{-1}{1296}$ 1st # $< \frac{1}{1000}$

Approx Sum $= \boxed{S_5 = .948}$