BC Calculus Improper Integrals Day 1 Notesheet **Name:** ____________________

To Begin Evaluate the following integrals using your calculator.

a) $\int_1^{100} \frac{1}{x} dx$ 4.605 **b)** $\int_1^{1000} \frac{1}{x} dx$ 6.908 **c)** $\int_1^{1000000} \frac{1}{x} dx$ 13.816

Based on the results you obtained above, predict the value of the following integral.

$\int_1^{\infty} \frac{1}{x} dx$ ∞ (Divergent)

Evaluate the following integrals using your calculator.

d) $\int_1^{100} e^{-x} dx$ .368 **e)** $\int_1^{1000} e^{-x} dx$ .368

Based on the results you obtained above, predict the value of the following integral.

$\int_1^{\infty} e^{-x} dx$ .368

Use Desmos to investigate the graphs of $f(x) = \frac{1}{x}$ and $g(x) = e^{-x}$ in the first quadrants. Why would these two integrals give such different results?

e^{-x} is approaching the x-axis much quicker

Improper Integrals

Integrals such as $\int_a^{\infty} f(x)dx$, $\int_{-\infty}^{b} f(x)dx$, and $\int_{-\infty}^{\infty} f(x)dx$ are called improper integrals. Integrals are classified as improper for three reasons.

1. They have an infinite interval of integration.
2. They have a discontinuity on the interior of the interval of integration.
3. Both 1 and 2.

Dealing with Infinite Intervals of Integration

If $\int_a^b f(x)dx$ exists for every $b > a$, then $\int_a^{\infty} f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$, provided the limit exists.

If $\int_a^b f(x)dx$ exists for every $a < b$, then $\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx$, provided the limit exists.

Basically, substitute a variable for ∞ or -∞ and find the limit as the variable approaches ∞ or -∞.

Example 1 Evaluate the following integrals.

a) $\int_1^{\infty} \frac{1}{x^{3/2}} dx$

$\lim_{b\to\infty} \int_1^b \frac{1}{x^{3/2}} dx$

$\int_1^b x^{-\frac{3}{2}} dx$

$-2x^{-\frac{1}{2}}\Big|_1^b$

$-2b^{-\frac{1}{2}} + 2(1)^{-\frac{1}{2}}$

$\lim_{b\to\infty} \cancel{\frac{-2}{\sqrt{b}}} + 2$

0

$= 2$

b) $\int_1^{\infty} \frac{1}{x^2} dx$

$\lim_{b\to\infty} \int_1^b \frac{1}{x^2} dx$

$\int_1^b x^{-2} dx$

$\lim_{b\to\infty} -1x^{-1}\Big|_1^b$

$-1 \cdot b^{-1} + 1(1)^{-1}$

$\lim_{b\to\infty} \frac{-1}{b} + 1 \rightarrow 1$

c) $\int_1^\infty \frac{1}{x^3}\,dx$ $\lim_{b\to\infty} -\frac{1}{2}x^{-2}\Big|_1^b$

Do you see a pattern emerging? What is it?

$\lim_{b\to\infty}\int_1^b \frac{1}{x^3}\,dx$ $\lim_{b\to\infty} -\frac{1}{2}b^{-2}+\frac{1}{2}(1)^{-2}$

$\int_1^b x^{-3}\,dx$ $\lim_{b\to\infty} \frac{-1}{2b^2}+\frac{1}{2} \to \frac{1}{2}$

P-Series Integrals

If $a > 0$, then $\int_a^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and divergent if $p \le 1$. These are called **p-series integrals**.

If $a = 1$ and $p > 1$, then $\int_a^\infty \frac{1}{x^p}\,dx$ converges to $\frac{1}{p-1}$.

Example 2 Evaluate the following integrals.

a) $\int_1^\infty \frac{1}{x^{2/3}}\,dx$ p-series $p=\frac{2}{3}$ $\frac{2}{3}<1$ divergent

b) $\int_1^\infty \frac{1}{x^{1.1}}\,dx$ p-series $p=1.1$ $1.1>1$ $=\frac{1}{1.1-1}=\frac{1}{.1}=\boxed{10}$

c) $\int_1^\infty x^{-7}\,dx$ $p=7$ $7>1$ $=\int_1^\infty \frac{1}{x^7}\,dx$ $\frac{1}{7-1}=\frac{1}{6}$

d) $\int_1^\infty 3\cdot x^{-3/2}\,dx$ $p=\frac{3}{2}$ $\frac{3}{2}>1$ $3\int_1^\infty \frac{1}{x^{3/2}}\,dx$ $\frac{1}{\frac{3}{2}-1}=\frac{1}{\frac{1}{2}}=2$ $3\cdot 2=\boxed{6}$

Example 3 Evaluate the following integrals.

a) $\int_1^\infty e^{-x}\,dx$

$\lim_{b\to\infty}\int_1^b e^{-x}\,dx$ $-e^{-x}\Big|_1^b$ $\lim_{b\to\infty} -e^{-b}+e^{-1}$ $\lim_{b\to\infty} -\frac{1}{e^b}+\frac{1}{e} \to \frac{1}{e}$

b) $\int_1^\infty \frac{1}{x}\,dx$ p-series $p=1$ $1\le 1$

$\lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx \to \lim_{b\to\infty} \ln x\Big|_1^b$ $\lim_{b\to\infty} \ln b - \ln 1$ ∞

Example 4 Evaluate the following integral.

Domain $x<3$

$\int_{-\infty}^0 \frac{1}{\sqrt{3-x}}\,dx$ $\lim_{a\to-\infty}\int_a^0 (3-x)^{-\frac{1}{2}}\,dx \to -2(3-x)^{\frac{1}{2}}\Big|_a^0$

$\lim_{a\to-\infty}\int_a^0 \frac{1}{\sqrt{3-x}}\,dx$ $\lim_{a\to-\infty} -2(3-0)^{\frac{1}{2}}+2(3-a)^{\frac{1}{2}} \to -2\sqrt{3}+\infty$ $\boxed{\text{divergent}}$

Example 5 Let $f(x) = e^{-2x}$ for $0 \le x < \infty$ and let R be the unbounded region in the first quadrant below the graph of f. Find the volume of the solid generated when R is revolved around the x-axis.

$V=\pi\int_0^\infty (e^{-2x})^2\,dx$

$=\pi\lim_{b\to\infty}\int_0^b e^{-4x}\,dx$

$\pi\lim_{b\to\infty} -\frac{1}{4}e^{-4x}\Big|_0^b$

$\pi\lim_{b\to\infty} -\frac{1}{4}e^{-4b}+\frac{1}{4}e^0$

$\pi\lim_{b\to\infty} -\frac{1}{4e^{4b}}+\frac{1}{4}$

$\pi\left(0+\frac{1}{4}\right)=\pi\frac{1}{4}=\frac{\pi}{4}$