BC Calculus Improper Integrals Day 1 Notesheet **Name:** ____________________

To Begin Evaluate the following integrals using your calculator.

a) $\int_1^{100} \frac{1}{x}dx \approx 4.6052$ b) $\int_1^{1000} \frac{1}{x}dx$ 6.9078 c) $\int_1^{1000000} \frac{1}{x}dx$ 13.816

Based on the results you obtained above, predict the value of the following integral.

$\int_1^{\infty} \frac{1}{x}dx$ ∞

Evaluate the following integrals using your calculator.

d) $\int_1^{100} e^{-x}dx$ .36788 e) $\int_1^{1000} e^{-x}dx$ .36788

Based on the results you obtained above, predict the value of the following integral.

$\int_1^{\infty} e^{-x}dx$ .36788

Use Desmos to investigate the graphs of $f(x) = \frac{1}{x}$ and $g(x) = e^{-x}$ in the first quadrants. Why would these two integrals give such different results?

b/c $\frac{1}{x}$ does not approach the x-axis as quickly as e^{-x}

Improper Integrals

Integrals such as $\int_a^{\infty} f(x)dx$, $\int_{-\infty}^{b} f(x)dx$, and $\int_{-\infty}^{\infty} f(x)dx$ are called improper integrals. Integrals are classified as improper for three reasons.

1. They have an infinite interval of integration.
2. They have a discontinuity on the interior of the interval of integration.
3. Both 1 and 2.

Dealing with Infinite Intervals of Integration

If $\int_a^b f(x)dx$ exists for every $b > a$, then $\int_a^{\infty} f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$, provided the limit exists.

If $\int_a^b f(x)dx$ exists for every $a < b$, then $\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx$, provided the limit exists.

Basically, substitute a variable for ∞ or -∞ and find the limit as the variable approaches ∞ or -∞.

Example 1 Evaluate the following integrals.

a) $\int_1^{\infty} \frac{1}{x^{3/2}}dx$

$\lim_{b\to\infty} \int_1^b x^{-\frac{3}{2}}dx$

$\lim_{b\to\infty} -2x^{-\frac{1}{2}}\Big|_1^b$

$\lim_{b\to\infty} \left(-2b^{-\frac{1}{2}} + 2(1)^{-\frac{1}{2}}\right) \to 0+2 = \boxed{2}$

b) $\int_1^{\infty} \frac{1}{x^2}dx$

$\lim_{b\to\infty} \int_1^b x^{-2}dx$

$\lim_{b\to\infty} -\frac{1}{x}\Big|_1^b$

$\lim_{b\to\infty} \left(-\frac{1}{b} + \frac{1}{1}\right) \to 0+1 = 1$

c) $\int_1^\infty \frac{1}{x^3}\,dx$ $\quad \frac{1}{2}$

$\lim_{b\to\infty}\int_1^b x^{-3}\,dx$

$\lim_{b\to\infty} \frac{1}{-2}x^{-2}\Big|_1^b \to \lim_{b\to\infty}\left(-\frac{1}{2}\frac{1}{b^2}+\frac{1}{2}\frac{1}{1^2}\right)$

Do you see a pattern emerging? What is it?

P-Series Integrals

If $a > 0$, then $\int_a^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and divergent if $p \le 1$. These are called **p-series integrals**.

If $a = 1$ and $p > 1$, then $\int_a^\infty \frac{1}{x^p}\,dx$ converges to $\frac{1}{p-1}$.

Example 2 Evaluate the following integrals.

a) $\int_1^\infty \frac{1}{x^{2/3}}\,dx$ $\quad p=\frac{2}{3}$

$\lim_{b\to\infty}\int_1^b x^{-\frac{2}{3}}\,dx$

$\lim_{b\to\infty} 3x^{\frac{1}{3}}\Big|_1^b \to \lim_{b\to\infty}\left(3b^{\frac{1}{3}}-3(1)^{\frac{1}{3}}\right) \to \boxed{\infty}$

b) $\int_1^\infty \frac{1}{x^{1.1}}\,dx$ $\quad p=1.1$

converge since $p>1$

$\frac{1}{1.1-1}=\frac{1}{.1}$ $\boxed{=10}$

c) $\int_1^\infty x^{-7}\,dx$ $\quad p=7$

converge $p>1$

$\frac{1}{7-1}=1/6$

d) $\int_1^\infty 3\cdot x^{-3/2}\,dx$ $\quad p=\frac{3}{2}$

converge $p>1$

$3\left[\frac{1}{\frac{3}{2}-1}\right]=3(2)=\boxed{6}$

Example 3 Evaluate the following integrals.

a) $\int_1^\infty e^{-x}\,dx$

$\lim_{b\to\infty}\int_1^b e^{-x}\,dx$

$\lim_{b\to\infty} -e^{-x}\Big|_1^b \to \lim_{b\to\infty}\left(-e^{-b}+e^{-1}\right)$

$= e^{-1}$

b) $\int_1^\infty \frac{1}{x}\,dx$ $\quad$ p-series $p=1$

$\boxed{\text{divergent}}$

Example 4 Evaluate the following integral.

$\int_{-\infty}^0 \frac{1}{\sqrt{3-x}}\,dx$ $\quad \lim_{b\to-\infty}\int_b^0 \frac{1}{\sqrt{3-x}}\,dx$

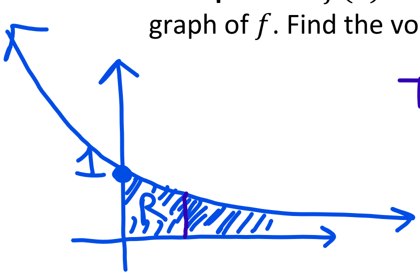
$\lim_{b\to-\infty}\int_b^0 (3-x)^{-\frac{1}{2}}\,dx$

$\lim_{b\to-\infty} -2(3-x)^{\frac{1}{2}}\Big|_b^0$

$\lim_{b\to-\infty}\left[-2(3-0)^{\frac{1}{2}}+2(3-b)^{\frac{1}{2}}\right]$

$\to \infty$ (diverge)

Example 5 Let $f(x) = e^{-2x}$ for $0 \le x < \infty$ and let R be the unbounded region in the first quadrant below the graph of f. Find the volume of the solid generated when R is revolved around the x-axis.



$\pi\int_0^\infty \left(e^{-2x}\right)^2\,dx$

$\pi\lim_{b\to\infty}\int_0^b e^{-4x}\,dx$

$\pi\lim_{b\to\infty} -\frac{1}{4}e^{-4x}\Big|_0^b$

$\pi\lim_{b\to\infty}\left(-\frac{1}{4}e^{-4b}+\frac{1}{4}e^{0}\right)$

$=\frac{\pi}{4}$