BC Calculus Improper Integrals Day 2 Notesheet **Name:** ______________________________

As we saw last time, improper integrals are evaluated by rewriting the integral as a proper integral and using limits. Not every improper integral equals a finite number. In fact, you would probably expect anything integrated to infinity or from negative infinity to be infinite. An improper integral that equals a finite value is said to converge to a value. An improper integral that does not equal a finite number is said to diverge.

When determining if an integral converges or diverges, the following information is helpful:

Convergent + Convergent = Convergent
Divergent + Divergent = Divergent
($\infty + \infty$ **or** $-\infty - \infty$)

Divergent + Convergent = Divergent
Divergent – Divergent = Indeterminate form
($\infty - \infty$ **or** $-\infty + \infty$)

POD $x=0$, $x=-\frac{1}{5}$

Example 1 Evaluate the following integrals.

a) $\int_1^{\infty}\left[\frac{1}{x}+\frac{5}{1+5x}\right]dx$

$\int_1^{\infty}\frac{1}{x}dx+\int_1^{\infty}\frac{5}{1+5x}dx$

$\lim_{b\to\infty}\int_1^b\frac{1}{x}dx+\lim_{b\to\infty}\int_1^b\frac{5}{1+5x}dx$

$5\int\frac{1}{1+5x}dx$

$\ln|x|\Big|_1^b+\frac{5}{5}\ln|1+5x|\Big|_1^b$

$\lim_{b\to\infty}\left(\ln b-\ln 1+\ln(1+5b)-\ln 1\right)$

$\lim_{b\to\infty}\ln b+\ln(1+5b)$

$\infty+\infty$

$\boxed{\text{Divergent}}$

b) $\int_1^{\infty}\frac{dx}{2x-1}$

POD $x=\frac{1}{2}$

$\lim_{b\to\infty}\int_1^b\frac{1}{2x-1}dx$

$\lim_{b\to\infty}\frac{1}{2}\ln|2x-1|\Big|_1^b$

$\lim_{b\to\infty}\frac{1}{2}\ln|2b-1|-\frac{1}{2}\ln|2\cdot 1-1|$

$\infty-0$

$\boxed{=\text{Divergent}}$

Improper Integrals Where Both Bounds are Infinite

If $\int_{-\infty}^{c}f(x)dx$ and $\int_c^{\infty}f(x)dx$ are both convergent, then $\int_{-\infty}^{\infty}f(x)dx=\int_{-\infty}^{c}f(x)dx+\int_c^{\infty}f(x)dx$, where c is any number. Note as well that this requires both of the integrals to be convergent in order for this integral to also be convergent. If either of the two integrals is divergent, then so is this integral.

Example 2 Evaluate the following integrals.

a) $\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx \rightarrow \int_{-\infty}^{0}\frac{1}{1+x^2}dx+\int_0^{\infty}\frac{1}{1+x^2}dx$

$\lim_{a\to-\infty}\int_a^0\frac{1}{1+x^2}dx+\lim_{b\to\infty}\int_0^b\frac{1}{1+x^2}dx$

$\lim_{a\to-\infty}\tan^{-1}x\Big|_a^0+\lim_{b\to\infty}\tan^{-1}x\Big|_0^b$

$\lim_{a\to-\infty}\tan^{-1}(0)-\tan^{-1}(a)+\lim_{b\to\infty}\tan^{-1}(b)-\tan^{-1}(0)$

$=0-\left(-\frac{\pi}{2}\right)+\frac{\pi}{2}-0$

$=\frac{\pi}{2}+\frac{\pi}{2}=\pi$

b) $\int_{-\infty}^{\infty}xe^{-x^2}dx \rightarrow \int_{-\infty}^{0}xe^{-x^2}dx+\int_0^{\infty}xe^{-x^2}dx$

$\lim_{a\to-\infty}\int_a^0 xe^{-x^2}dx$
$+\lim_{b\to\infty}\int_0^b xe^{-x^2}dx$

$u=-x^2$
$\frac{du}{dx}=-2x$
$-\frac{1}{2}du=x\,dx$

$-\frac{1}{2}\int e^u\,du$
$-\frac{1}{2}e^u$
$-\frac{1}{2}e^{-x^2}$

$\lim_{a\to-\infty}-\frac{1}{2}e^{-x^2}\Big|_a^0+\lim_{b\to\infty}-\frac{1}{2}e^{-x^2}\Big|_0^b$

$\lim_{a\to-\infty}-\frac{1}{2}e^0+\frac{1}{2}e^{-a^2}+\lim_{b\to\infty}-\frac{1}{2}e^{-b^2}+\frac{1}{2}e^0$

$-\frac{1}{2}+0-0+\frac{1}{2}$

$\boxed{=0}$

① setup
② solve

c) Find the area of the region bounded by the graph of $y=\frac{20}{x^2+1}$ and the x-axis.

$\int_{-\infty}^{\infty}\frac{20}{x^2+1}dx=\int_{-\infty}^{0}\frac{20}{x^2+1}dx+\int_0^{\infty}\frac{20}{x^2+1}dx$

$\lim_{a\to-\infty}\int_a^0\frac{20}{x^2+1}dx+\lim_{b\to\infty}\int_0^b\frac{20}{x^2+1}dx \rightarrow$ Now look at a) ↑ so our answer is 20π

$=20\pi$

When an integral is improper has a finite interval of integration, it is improper because its interval spans an infinite discontinuity (vertical asymptote). These are harder to spot, so be vigilant.

→ seems to be approaching 1.5

Example 3 Approximate the value of the following integrals using your calculator.

a) $\int_{0.01}^{1} x^{-1/3}dx$ 1.43 b) $\int_{0.001}^{1} x^{-1/3}dx$ 1.485 c) $\int_{0.0001}^{1} x^{-1/3}dx$ 1.497

→ seems to be diverging

d) $\int_{0.01}^{1} x^{-3}dx$ 4999.5 e) $\int_{0.001}^{1} x^{-3}dx$ 499,999.5 f) $\int_{0.0001}^{1} x^{-3}dx$ 49,999,999

When we recognize an infinite discontinuity at an endpoint, we have to set up a one-sided limit. When the infinite discontinuity is on the interior, we have to set up two integrals, one approaching the vertical asymptote from each side.

POD $x=0$

Example 4 Evaluate $\int_0^1 \frac{1}{x^{1/3}}dx$

$$\lim_{a\to 0^+}\int_a^1 \frac{1}{x^{\frac{1}{3}}}dx \to \lim_{a\to 0^+}\int_a^1 x^{-\frac{1}{3}}dx$$

$$\lim_{a\to 0^+} \frac{3}{2}x^{\frac{2}{3}}\Big|_a^1 \to \lim_{a\to 0^+} \frac{3}{2}(1)^{\frac{2}{3}} - \frac{3}{2}a^{\frac{2}{3}}$$

$$\frac{3}{2} - 0 = \frac{3}{2}$$

P.O.D $x=0$

Example 5 Evaluate $\int_0^1 \frac{1}{x^3}dx$

$$\lim_{a\to 0^+}\int_a^1 x^{-3}dx \to \lim_{a\to 0^+} -\frac{1}{2}x^{-2}\Big|_a^1$$

$$\to \lim_{a\to 0^+} -\frac{1}{2}(1)^{-2} + \frac{1}{2}a^{-2} \to \lim_{a\to 0^+} -\frac{1}{2} + \frac{1}{2a^2}$$

$$-\frac{1}{2} + \infty$$

$\to \infty$ divergent

POD $x=27$

Example 6 Evaluate $\int_0^{27} \frac{dx}{\sqrt[3]{27-x}}$

$$\lim_{b\to 27^-}\int_0^b (27-x)^{-\frac{1}{3}}dx$$

$$\lim_{b\to 27^-} -\frac{3}{2}(27-x)^{\frac{2}{3}}\Big|_0^b$$

$$\lim_{b\to 27^-} -\frac{3}{2}(27-b)^{\frac{2}{3}} + \frac{3}{2}(27-0)^{\frac{2}{3}}$$

$$0 + \frac{3}{2}(27)^{\frac{2}{3}}$$

$$+\frac{3}{2}(3^3)^{\frac{2}{3}}$$

$$\frac{3}{2}3^2 = \frac{3}{2}9 = \frac{27}{2}$$

What if there is both an infinite bound and a discontinuity?

Example 7 Evaluate $\int_0^\infty \frac{1}{x^2}dx$

$$\int_0^1 \frac{1}{x^2}dx + \int_1^\infty \frac{1}{x^2}dx$$

$$\lim_{a\to 0^+}\int_a^1 \frac{1}{x^2}dx + \lim_{b\to\infty}\int_1^b \frac{1}{x^2}dx$$

$$\lim_{a\to 0^+}\int_a^1 x^{-2}dx + \lim_{b\to\infty}\int_1^b x^{-2}dx$$

$$\lim_{a\to 0^+} -x^{-1}\Big|_a^1 + \lim_{b\to\infty} -x^{-1}\Big|_1^b$$

$$\lim_{a\to 0^+} -\frac{1}{x}\Big|_a^1 + \lim_{b\to\infty} -\frac{1}{x}\Big|_1^b$$

$$\lim_{a\to 0^+} -\frac{1}{1} + \frac{1}{a} + \lim_{b\to\infty} -\frac{1}{b} + \frac{1}{1}$$

$$-1 + \infty + 0 + 1 \to \infty$$

divergent