BC Calculus Improper Integrals Day 3 Notesheet **Name:** ______________________________

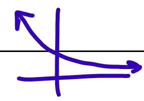
Sometimes, we cannot find the antiderivative of an integrand of an improper integral. However, we might be able to draw a conclusion about its convergence or divergence if we can compare it to something similar for which we do something.

Direct Comparison Test for Convergence or Divergence

Let f and g be continuous on $[a, \infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$, then

If $\int_a^\infty g(x)dx$ converges, then $\int_a^\infty f(x)dx$ also converges. In other words, if the larger one converges, the smaller one also converges.

If $\int_a^\infty f(x)dx$ diverges, then $\int_a^\infty g(x)dx$ also diverges. In other words, if the smaller one diverges, the larger one also diverges.



Example 1 If $0 \le e^{-x^2} \le e^{-x}$ for all $x \ge 1$, determine if $\int_1^\infty e^{-x^2}dx$ converges or diverges.

$$\int_1^\infty e^{-x}dx \to \lim_{b\to\infty}\int_1^b e^{-x}dx \to \lim_{b\to\infty} -e^{-x}\Big|_1^b \to \lim_{b\to\infty} -e^{-b}+e^{-1}$$

$$\downarrow$$

$$0+e^{-1}$$

converge

by DCT, $\int_1^\infty e^{-x^2}dx$ converges also

Example 2 Determine if $\int_\pi^\infty \frac{2+\cos x}{x}dx$ converges or diverges.

$$0 \le \frac{1}{x} \le \frac{2+\cos x}{x} \quad \text{for } x \ge \pi$$

$\int_\pi^\infty \frac{1}{x}dx$ p-series $p=1$, $1 \le 1$ so divergent

by DCT $\int_\pi^\infty \frac{2+\cos x}{x}dx$ diverges

Example 3 Determine if $\int_4^\infty \frac{2}{x+e^x}dx$ converges or diverges.

$\frac{2}{x}$ or $\frac{2}{e^x}$

small $\to \frac{2}{x+e^x}$

prefer to compare to something that converges b/c if we compare to $\frac{2}{x}$ which diverges, it's inconclusive

$$0 \le \frac{2}{x+e^x} \le \frac{2}{e^x} \quad \text{for } x \ge 4$$

$$\int_4^\infty \frac{2}{e^x}dx \to \lim_{b\to\infty}\int_4^b 2e^{-x}dx$$

$$\lim_{b\to\infty} -2e^{-x}\Big|_4^b \to \lim_{b\to\infty} -2e^{-b}+2e^{-4}$$

converges

by DCT $\int_4^\infty \frac{2}{x+e^x}dx$ converges

Example 4 Determine if $\int_4^\infty \frac{2}{x-e^{-x}}\,dx$ converges or diverges.

$\frac{2}{x}$ or $\cancel{\frac{2}{-e^{-x}}}$

$\int_4^\infty \frac{2}{x}\,dx$ p-series $p=1$, $1 \le 1$ diverge

$0 \le \frac{2}{x} \le \frac{2}{x-e^{-x}}$ for $x \ge 4$

by DCT $\int_4^\infty \frac{2}{x-e^{-x}}\,dx$ diverges

Limit Comparison Test for Convergence or Divergence

If the positive functions f and g are continuous on $[a,\infty)$ and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, 0 < L < \infty,$$

if the end behavior of the ratio of the 2 functions is a finite positive number

then

$\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ both converge or diverge.

$\lim_{b\to\infty} \frac{\frac{2}{x+e^x}}{\frac{2}{x}} \to \frac{2}{x+e^x}\cdot\frac{x}{2}$

$\frac{x}{x+e^x} \to 0$

Example 5 Determine if $\int_4^\infty \frac{2}{x+e^x}\,dx$ converges or diverges.

$\lim_{x\to\infty} \frac{\frac{2}{x+e^x}}{\frac{2}{e^x}} \to \lim_{b\to\infty} \frac{2}{x+e^x}\cdot\frac{e^x}{2} \to 1$

1 is a positive constant ✓ $0<1<\infty$

since $\int_4^\infty \frac{2}{e^x}\,dx$ converges by LCT $\int_4^\infty \frac{2}{x+e^x}\,dx$ converges

Example 6 Determine if $\int_1^\infty \frac{dx}{2+x^2}$ converges or diverges using the Limit Comparison Test

$\lim_{x\to\infty} \frac{\frac{1}{2+x^2}}{\frac{1}{x^2}} \to 1$

$0<1<\infty$ ✓

$\int_1^\infty \frac{1}{x^2}\,dx$ p-series $p=2$, $2>1 \to$ converges

by LCT $\int_1^\infty \frac{1}{1+x^2}dx$ converges

Example 7 Determine if $\int_1^\infty \frac{dx}{5+x^3}$ converges or diverges using the Limit Comparison Test

$\lim_{x\to\infty} \frac{\frac{1}{5+x^3}}{\frac{1}{x^3}} \to 1$

$0<1<\infty$

$\int_1^\infty \frac{1}{x^3}\,dx$ p-series $p=3$, $3>1 \to$ converges

by LCT $\int_1^\infty \frac{1}{5+x^3}dx$ converges