BC Calculus Improper Integrals Day 3 Notesheet **Name:** ____________________

Sometimes, we cannot find the antiderivative of an integrand of an improper integral. However, we might be able to draw a conclusion about its convergence or divergence if we can compare it to something similar for which we do something.

Direct Comparison Test for Convergence or Divergence

Let f and g be continuous on $[a, \infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$, then

If $\int_a^\infty g(x)dx$ converges, then $\int_a^\infty f(x)dx$ also converges. In other words, if the larger one converges, the smaller one also converges.

If $\int_a^\infty f(x)dx$ diverges, then $\int_a^\infty g(x)dx$ also diverges. In other words, if the smaller one diverges, the larger one also diverges.

Example 1 If $0 \le e^{-x^2} \le e^{-x}$ for all $x \ge 1$, determine if $\int_1^\infty e^{-x^2}dx$ converges or diverges.

$e^{-x} \to \frac{1}{e^x}$ $\quad e^{-x^2} \to \frac{1}{e^{x^2}}$

$\lim_{b\to\infty} \int_1^b e^{-x}dx$

$\lim_{b\to\infty} -e^{-x}\Big|_1^b \to \lim_{b\to\infty} -e^{-b} + e^{-1} \to \lim_{b\to\infty} \frac{1}{-e^b} + \frac{1}{e}$

$0 + \frac{1}{e} = \frac{1}{e}$

therefore $\int_1^\infty e^{-x^2}dx$ also converges by DCT

Example 2 Determine if $\int_\pi^\infty \frac{2+\cos x}{x}dx$ converges or diverges.

$\frac{1}{x} \le \frac{2+\cos x}{x}$ for $x \ge \pi$

$\lim_{b\to\infty} \int_\pi^b \frac{1}{x}dx \to$ p-series $p = 1$, $1 \le 1$ ∴ this diverges

so by DCT $\int_\pi^\infty \frac{2+\cos x}{x}dx$ diverges

Example 3 Determine if $\int_4^\infty \frac{2}{x+e^x}dx$ converges or diverges.

$\frac{2}{x+e^x} \le \frac{2}{x}$ for $x \ge 4$

If $\lim_{b\to\infty} \int_4^b \frac{2}{x}dx$ p-series $p=1$, $1 \le 1$ divergent

But the larger f(x) diverging inconclusive

$\frac{2}{x+e^x} \le \frac{2}{e^x}$ for $x \ge 4$

$\lim_{b\to\infty} \int_4^b \frac{2}{e^x}dx \to \lim_{b\to\infty} \int_4^b 2e^{-x}dx$

$-2e^{-x}\Big|_4^b \to -2e^{-b} + 2e^{-4}$

$\lim_{b\to\infty} \frac{-2}{e^b} + \frac{2}{e^4} \to$ converges

therefore by DCT $\int_4^\infty \frac{2}{x+e^x}dx$ also converges

Example 4 Determine if $\int_4^\infty \frac{2}{x-e^{-x}}\,dx$ converges or diverges.

$\frac{2}{x-e^{-x}} \geq \frac{2}{x}$ for $x \geq 4$

$\lim_{b\to\infty} \int_4^b \frac{2}{x}\,dx$ p-series $p=1$ $1 \leq 1$ $\therefore$ divergent

by DCT $\int_4^\infty \frac{2}{x-e^{-x}}\,dx$ diverges

Limit Comparison Test for Convergence or Divergence
If the positive functions f and g are continuous on $[a,\infty)$ and if $$\lim_{x\to\infty}\frac{f(x)}{g(x)} = L, 0 < L < \infty,$$ then $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ both converge or diverge.

Example 5 Determine if $\int_4^\infty \frac{2}{x+e^x}\,dx$ converges or diverges.

$$\lim_{x\to\infty} \frac{\frac{2}{x+e^x}}{\frac{2}{e^x}} \to \lim_{x\to\infty} \frac{2}{x+e^x}\cdot\frac{e^x}{2}$$

$$\lim_{x\to\infty} \frac{2e^x}{2e^x+2x} \to 1$$

$0<1<\infty$

since $\int_4^\infty \frac{2}{e^x}\,dx$ converges by LCT $\int_4^\infty \frac{2}{x+e^x}\,dx$ converges

Example 6 Determine if $\int_1^\infty \frac{dx}{2+x^2}$ converges or diverges using the Limit Comparison Test

$$\lim_{x\to\infty} \frac{\frac{1}{2+x^2}}{\frac{1}{x^2}} \to \lim_{x\to\infty} \frac{1}{2+x^2}\cdot\frac{x^2}{1} = 1$$ and $0<1<\infty$

since $\int_1^\infty \frac{1}{x^2}\,dx$ converges by LCT $\int_1^\infty \frac{1}{2+x^2}\,dx$ converge

p-series $p=2$ $2>1$

Example 7 Determine if $\int_1^\infty \frac{dx}{5+x^3}$ converges or diverges using the Limit Comparison Test

$$\lim_{x\to\infty} \frac{\frac{1}{5+x^3}}{\frac{1}{x^3}} = 1$$ and $0<1<\infty$

since $\int_1^\infty \frac{1}{x^3}\,dx$ converges by LCT $\int_1^\infty \frac{1}{5+x^3}\,dx$ converge

p-series $p=3$ $3>1$