BC Calculus Integration by Parts Notesheet **Name:** ___________________________________

When you integrate, you find the antiderivative, or you "undo" the derivative. When you used u-substitution, you were undoing a derivative that involved the Chain Rule. Today, we will learn how to undo a derivative that involves the product rule.

First, recall the way we differentiate functions using the Product Rule and differential notation.

$$(uv)' = udv + vdu$$

Using the Product Rule, we can develop a formula for integration by parts

$$\int (uv)' = \int u\,dv + \int v\,du$$

$$uv = \int u\,dv + \int v\,du$$

$$uv - \int v\,du = \int u\,dv$$

Integration by Parts

If u and v are functions of x and have continuous derivatives, then

$$\boxed{\int u\,dv = uv - \int v\,du}$$

"Oohvee vadoo"

The trick with integration by parts is determining which function to select as your "u." When selecting u, think of the word LIPET → this helps us figure what to set $u =$

Logarithmic **I**nverse Trig **P**olynomial **E**xponential **T**rigonometric

Example 1 Evaluate $\int x \cos x\,dx$

$= \int u\,dv$

$u = x \qquad \int dv = \int \cos x\,dx$

$du = dx \qquad v = \sin x$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx$$

$$\boxed{= x\sin x + \cos x + c}$$

Example 2 Evaluate $\int 2xe^{4x}\,dx$

$\int u\,dv = uv - \int v\,du$

$u = 2x \qquad \int dv = \int e^{4x}dx$

$du = 2dx, \qquad v = \frac{1}{4}e^{4x}$

$$\int 2xe^{4x}dx = (2x)\left(\tfrac{1}{4}e^{4x}\right) - \int \tfrac{1}{4}e^{4x}\,2\,dx$$

$$= \tfrac{1}{2}xe^{4x} - \tfrac{1}{2}\cdot\tfrac{1}{4}e^{4x} + c$$

$$\boxed{= \tfrac{1}{2}xe^{4x} - \tfrac{1}{8}e^{4x} + c}$$

Example 3 Evaluate $\int x \ln x\,dx$

$u = \ln x \qquad \int dv = \int x\,dx$

$du = \frac{1}{x}dx \qquad v = \frac{1}{2}x^2$

$$\int x\ln x\,dx = (\ln x)\left(\tfrac{1}{2}x^2\right) - \int \tfrac{1}{2}x^2\,\tfrac{1}{x}\,dx$$

$$= \tfrac{1}{2}x^2\ln x - \tfrac{1}{2}\int x\,dx \rightarrow \boxed{\tfrac{1}{2}x^2\ln x - \tfrac{1}{2}\,\tfrac{x^2}{2} + c}$$

Example 4 Evaluate $\int x \sec^2 x\,dx$

Final answer: $\boxed{x\tan x + \ln|\cos x| + c}$

$u = x \qquad \int dv = \int \sec^2 x\,dx$

$du = dx \qquad v = \tan x$

$$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx$$

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx \rightarrow u = \cos x,\ du = -\sin x\,dx$$

$$= -\int \frac{1}{u}\,du \rightarrow -\ln|u| + c$$

$$\rightarrow -\ln|\cos x| + c$$

Sometimes, an integration by parts problem will try to disguise itself.

Example 5 Evaluate $\int \ln x\,dx$

$u = \ln x \qquad \int dv = \int dx$

$du = \frac{1}{x}dx \qquad v = x$

$$\int \ln x\,dx = (\ln x)(x) - \int x\,\tfrac{1}{x}\,dx$$

$$= x\ln x - \int dx$$

$$\boxed{= x\ln x - x + c}$$

Example 6 Evaluate $\int \arctan x\,dx$

$u = \arctan x \qquad \int dv = \int dx$

$du = \frac{1}{1+x^2}dx \qquad v = x$

Final answer: $\boxed{x\arctan x - \tfrac{1}{2}\ln|1+x^2| + c}$

$$\int \arctan x\,dx = (\arctan x)(x) - \int x\,\frac{1}{1+x^2}dx$$

$u = 1 + x^2$

$\frac{du}{dx} = 2x \rightarrow \frac{1}{2}du = x\,dx$

$$\rightarrow \tfrac{1}{2}\int \frac{1}{u}\,du \rightarrow \tfrac{1}{2}\ln|u| + c$$

$$\rightarrow \tfrac{1}{2}\ln|1+x^2| + c$$

Sometimes, we have to repeat ourselves. Sometimes, we have to repeat ourselves.

LIPET

Example 7 Evaluate $\int x^2 \sin x \, dx$

$u = x^2 \quad \int dv = \int \sin x \, dx$

$du = 2x\,dx \quad v = -\cos x$

$= (x^2)(-\cos x) - \int -\cos x \cdot 2x \, dx$

$= -x^2 \cos x + 2\int x \cos x \, dx$

$u = x \quad \int dv = \int \cos x \, dx$

$du = dx \quad v = \sin x$

$x \sin x - \int \sin x \, dx$

$x \sin x + \cos x + c$

Final $\boxed{-x^2 \cos x + 2x \sin x + 2\cos x + c}$

We have seen that integrals of the form $\int f(x)g(x)dx$, in which f can be differentiated repeatedly to become zero and g can be integrated repeatedly without difficulty, are natural candidates for integration by parts. However, if many repetitions are required, the calculations can become cumbersome. In situations like this, using a table can help organize the calculations and speed the process up.

Example 8 Evaluate $\int x^2 \sin x \, dx$ using the tabular method

u (derive)		dv (integrate)
x^2	$+$	$\sin x$
$2x$	$-$	$-\cos x$
2	$+$	$-\sin x$
0		$\cos x$

$= \boxed{-x^2 \cos x + 2x \sin x + 2\cos x + c}$

Example 9 Evaluate $\int t^4 e^{2t} \, dt$

u (derive)		dv (integrate)
t^4	$+$	$e^{2t}dt$
$4t^3$	$-$	$\frac{1}{2}e^{2t}$
$12t^2$	$+$	$\frac{1}{4}e^{2t}$
$24t$	$-$	$\frac{1}{8}e^{2t}$
24	$+$	$\frac{1}{16}e^{2t}$
0		$\frac{1}{32}e^{2t}$

$$\boxed{\frac{1}{2}t^4 e^{2t} - t^3 e^{2t} + \frac{3}{2}t^2 e^{2t} - \frac{3}{2}t e^{2t} + \frac{3}{4}e^{2t} + c}$$

Example 10 Evaluate $\int x^3 \cos(2x) \, dx$

u		dv
x^3	$+$	$\cos(2x)$
$3x^2$	$-$	$\frac{1}{2}\sin(2x)$
$6x$	$+$	$-\frac{1}{4}\cos(2x)$
6	$-$	$-\frac{1}{8}\sin(2x)$
0		$\frac{1}{16}\cos(2x)$

$$\boxed{\frac{1}{2}x^3 \sin(2x) + \frac{3}{4}x^2 \cos(2x) - \frac{3}{4}x \sin(2x) - \frac{3}{8}\cos(2x) + c}$$

What if neither function goes to zero?

LIPET

Example 11 Evaluate $\int e^x \cos(x) \, dx$

$u = e^x \quad \int dv = \int \cos x \, dx$

$du = e^x dx \quad v = \sin x$

$= e^x \sin x - \int \sin x \, e^x \, dx$

$\int \sin x \, e^x \, dx$

$u = e^x \quad \int dv = \int \sin x \, dx$

$du = e^x dx \quad v = -\cos x$

$-e^x \cos x + \int \cos x \, e^x \, dx$

original: $\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int \cos x \, e^x \, dx$

$+\int e^x \cos x \, dx \qquad +\int \cos x \, e^x \, dx$

$$\frac{2\int e^x \cos x \, dx}{2} = \frac{e^x \sin x + e^x \cos x}{2}$$

$$\boxed{= \frac{e^x \sin x + e^x \cos x}{2}}$$