**BC Calculus Intro to Power Series Notesheet** **Name: ________________________________________**

**Definition of Power Series**

If $x$ is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

is called a **power series** where $x$ is a variable and $a_n$'s are coefficients of the series.

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots$$

Is a power series **centered at** $c$, where $c$ is a constant, called the **center**.

A power series may converge for some values of $x$ and diverge for other values of $x$. The sum of the series is a function

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

Whose domain is the set of all $x$ for which the series converges. Notice how $f(x)$ resembles a polynomial. The only difference is that $f(x)$ above has infinitely many terms. For instance, if we say $a_n = 1$ ffor all $n$, then the power series becomes a geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

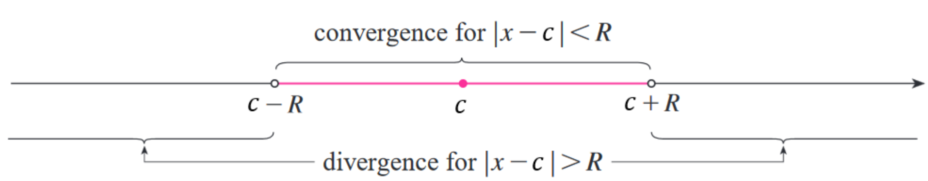
Where $r = x$ (you are multiplying by $x$ each time) and the series will converge when $|x| < 1$ and diverge when $|x| \geq 1$ by the geometric series test. Therefore, the domain of the function would be $-1 < x < 1$ or $(-1, 1)$ in interval notation.

The set of x-values for which the power series converges is called the **interval of convergence**. The distance from the center to the outside of the interval of convergence is called the radius of convergence and is denoted by $R$. In the example above, the center was 0 and the outside of the interval was 1, so $R = 1 - 0 = 1$. The power series above will converge as long as $x$ is less than one unit from the center of the series.

**Possibilities for the Interval and Radius of Convergence of a Power Series**

For a power series centered at $c$, one of the following will occur:

1. The series converges only at $c$ (All power series converge at the center!). In this situation, $R = 0$.
2. The series converges for all $x$. In this situation, $R = \infty$.
3. The series converges for $|x - c| < R$ and diverges for $|x - c| > R$. In this situation, the series will converge over the domain $(c - R, c + R)$. This idea is shown graphically below.




4. The endpoints if the interval of convergence may or may not be included in the domain. To find out if they are, we have to test them individually.

**Note:** To find the radius/interval of convergence, we use the ratio test (or the root test on rare occasion).

**Example 1** Find the radius and interval of convergence for each of the following power series.

a) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| \to \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| \qquad \lim_{n\to\infty} \left| \frac{\cancel{x^n} \cdot x^1}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{x^n}} \right|$$

$$\lim_{n\to\infty} \frac{x}{n+1} = 0$$

converge by ratio test for all values of $x$

R $\infty$ IOC $(-\infty, \infty)$

b) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-5)^n}{n2^n}$

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+2}(x-5)^{n+1}}{(n+1)\,2^{n+1}}}{\frac{(-1)^{n+1}(x-5)^n}{n\,2^n}} \right| \to \lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+2}}\cancel{(x-5)^n}(x-5)^1}{(n+1)\cancel{2^n}\,2} \cdot \frac{n\,\cancel{2^n}}{\cancel{(-1)^{n+1}}\cancel{(x-5)^n}} \right|$$

EBM 1

$$\left|\frac{x-5}{2}\right| < 1$$

$$|x-5| < 2$$

$\boxed{\text{C } 5 \quad \text{R } 2}$

$\boxed{\text{IOC } 3 < x \le 7}$

$\boxed{\begin{array}{l} \text{Check endpts} \quad \text{diverge} \\ x = 3 \quad \frac{(-1)^{2n+1}}{n} \to \text{always } - \\ \qquad \to \text{Harmonic} \end{array}}$

$\boxed{\begin{array}{l} \text{Check} \quad \frac{(-1)^{n+1}}{n} \to \text{converges by AST} \\ x = 7 \end{array}}$

c) $\sum_{n=0}^{\infty} n!\,(x-3)^n$

$$\lim_{n\to\infty} \left| \frac{(n+1)!\,(x-3)^{n+1}}{n!\,(x-3)^n} \right| \to \lim_{n\to\infty} \left| (n+1)(x-3) \right| \to \infty$$

$\infty > 1$ diverges by ratio test

C 3 $\quad \boxed{\text{R } 0} \quad \boxed{\text{IOC } x = 3}$

d) $\sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$

$$\lim_{n\to\infty} \left| \frac{\left(\frac{x}{3}\right)^{n+1}}{\left(\frac{x}{3}\right)^n} \right| \to \lim_{n\to\infty} \left| \frac{x}{3} \right|$$

$$\left|\frac{x}{3}\right| < 1$$

$$|x| < 3$$

$\boxed{\begin{array}{l} \text{center } 0 \\ \text{R } 3 \\ \text{IOC } -3 < x < 3 \end{array}}$

Check endpts $x = -3$ $\left(\frac{-3}{3}\right)^n = (-1)^n$ oscillating $1 \leftrightarrow -1$ diver

$x = 3$ $\left(\frac{3}{3}\right)^n = (1)^n$ diverges

e) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!}}{\frac{(-1)^n x^{2n+1}}{(2n+1)!}} \right| \qquad \lim_{n\to\infty} \left| \frac{(-1)^{n+1}\cancel{x^{2n}}\,x^{3\,2}}{(2n+3)(2n+2)\cancel{(2n+1)!}} \cdot \frac{\cancel{(2n+1)!}}{(-1)^n\,\cancel{x^{2n}}\,\cancel{x^1}} \right|$$

$$\lim_{n\to\infty} \left| \frac{x^2}{(2n+3)(2n+2)} \right| = 0$$

converges for all $x$

R. $\infty$ I.O.C. $(-\infty, \infty)$