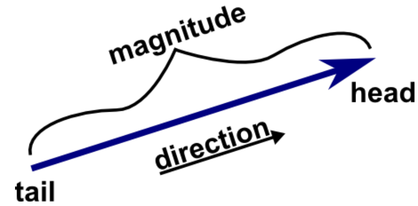
BC Calculus Intro to Vectors Notesheet **Name:** ____

In some quantities we deal with, only the magnitude (value or number) is important. Your speedometer tells you how fast you are going. These are called **scalars**. If we are concerned about both the magnitude and direction, then we have a quantity called a **vector**. To reiterate, with a scalar, only magnitude is important. With a vector, both magnitude and direction are important.

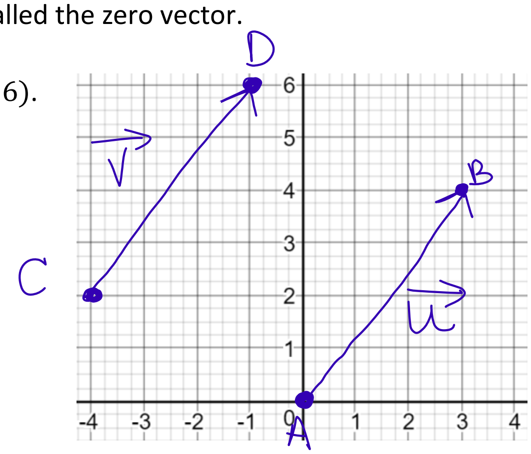


There are many ways to represent a vector. We will learn several. The first is a geometric interpretation, an example of which is shown above. In the geometric interpretation of a vector, we use a directed line segment to represent the vector. The starting point of the vector is called the tail or initial point, and the ending point of the vector, indicated with an arrow, is called the head or terminal point. The length of the vector is the vector's magnitude, and the direction the vector points is the vector's direction. Two **vectors are equal** if they have the **same magnitude and direction**.

Component Form of a Vector

If $\mathbf{v}$ is a vector in the plane equal to the vector with initial point $(0,0)$ and terminal point (v_1, v_2), then the component form of $\mathbf{v}$ is $v = \langle v_1, v_2 \rangle$ where numbers v_1 and v_2 are components of vector $\mathbf{v}$. v_1 is the horizontal component and v_2 is the vertical component. The vector $\langle 0,0 \rangle$ is called the zero vector.

Example 1 Let $A = (0,0)$, $B = (3,4)$, $C = (-4,2)$, and $D = (-1,6)$. Show that the vectors $\mathbf{u} = \overrightarrow{AB}$ and $\mathbf{v} = \overrightarrow{CD}$ are equal.



$\vec{u} = \langle 3-0, 4-0 \rangle = \langle 3, 4 \rangle$

$\vec{v} = \langle -1--4, 6-2 \rangle = \langle 3, 4 \rangle$

Vector Notation

There are a few different ways vectors are represented using notation. One uses the terminal and initial points such as $\overrightarrow{AB}$. This notation indicates that the vector starts at point A and terminates at point B. Another notation requires the naming of a vector such as $\vec{u}$. A third way is similar to the previous way, but instead of using an arrow, bold print is used. For example, $\mathbf{u}$ would mean the vector named u. The notation $\boxed{|\overrightarrow{AB}| \text{ or } \|\overrightarrow{AB}\|}$ represents the length or magnitude of the vector $\overrightarrow{AB}$.

↳magnitude

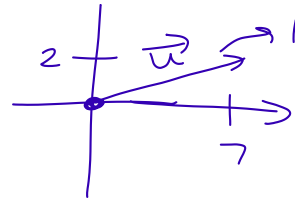
The location of a vector does not matter. If the vectors have the same length and same direction, then they are equal no matter where they are located on the coordinate plane. A vector with initial point at the origin is said to be in standard position. In example 1, $\overrightarrow{AB}$ is in standard position.

Magnitude of a Vector

The magnitude or length of the vector $\mathbf{v} = \overrightarrow{AB}$ determined by $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$|\overrightarrow{AB}| = \|\overrightarrow{AB}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(v_1)^2 + (v_2)^2}$$

e.g



length of $\vec{u}$ is $\sqrt{53}$

Example 2 Find the magnitude of $\mathbf{u} = \langle 7, 2 \rangle$.

$|\vec{v}| = \sqrt{49 + 4} = \sqrt{53}$

Example 3 Find the magnitude of $\overrightarrow{AB}$ if $A = (1,2)$ and $B = (5, -1)$

$\overrightarrow{AB} = \langle 5-1, -1-2 \rangle$

$= \langle 4, -3 \rangle$

$\boxed{|\overrightarrow{AB}| = 5}$

Unit Vectors

A unit vector is a vector of whose magnitude is 1. To find a unit vector in the direction of a given vector, divide each component of the vector by the magnitude of the vector. For example, if the vector $\mathbf{v}$ is not the zero vector, then the unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

start / end

Example 4 Find the component form of the vector with initial point $P(-3, 4)$ and terminal point $Q(-5, 2)$, the length of the vector, and a unit vector in the direction of the vector from P to Q.

$\overrightarrow{PQ} = \langle -5 - -3, 2 - 4 \rangle$

$\overrightarrow{PQ} = \langle -2, -2 \rangle$ or $-2i - 2j$

$|\overrightarrow{PQ}| = \sqrt{2^2 + 2^2} = \sqrt{8}$ or $2\sqrt{2}$

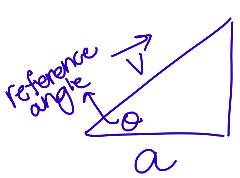
$\frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \langle \frac{-2}{2\sqrt{2}}, \frac{-2}{2\sqrt{2}} \rangle$

$\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$

Finding the component form of a vector with a given length and direction

Horizontal / vertical

In general, for vector $\mathbf{v} = \langle a, b \rangle$, $a = |\mathbf{v}| \cos\theta$ and $b = |\mathbf{v}| \sin\theta$. These come directly from right-triangle trigonometry.



$\cos\theta = \frac{a}{|v|}$ $\quad$ $\sin\theta = \frac{b}{|v|}$

$|v|\cos\theta = a$ $\quad$ $|v|\sin\theta = b$

Example 5 Find the component form of the vector $\mathbf{v}$ of length 3 that makes an angle of $\frac{\pi}{4}$ with the positive x-axis.

$\vec{v} = \langle 3\cos\frac{\pi}{4}, 3\sin\frac{\pi}{4} \rangle$

$= \langle \frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2} \rangle$

Operations Using Vectors

Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ be vectors and k be a real number scalar.

$\mathbf{0} = \langle 0, 0 \rangle$	The zero vector. Both components are 0.
$\mathbf{u} + \mathbf{v} = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle = \langle u_1 + v_1, u_2 + v_2 \rangle$	Add the x components and add the y components
$\mathbf{u} - \mathbf{v} = \langle u_1, u_2 \rangle - \langle v_1, v_2 \rangle = \langle u_1 - v_1, u_2 - v_2 \rangle$	Subtract the x components and subtract the y components.
$k\mathbf{v} = k\langle v_1, v_2 \rangle = \langle kv_1, kv_2 \rangle$	Multiply each component by the scalar

If $k = -1$, it reverses the direction of the vector, resulting in the opposite vector of the original.

Whenever you add or subtract vectors, the result is called the **resultant vector**.

Example 6 Let $\mathbf{u} = \langle 1, 3 \rangle$ and $\mathbf{v} = \langle 4, 7 \rangle$. Find the following

a) $2\mathbf{u} + \mathbf{v}$

$2\langle 1,3 \rangle + \langle 4,7 \rangle$

$\langle 2,6 \rangle + \langle 4,7 \rangle$

$\boxed{\langle 6,13 \rangle}$

b) $\mathbf{u} - \mathbf{v}$

$\langle 1,3 \rangle - \langle 4,7 \rangle$

$\boxed{\langle -3, -4 \rangle}$

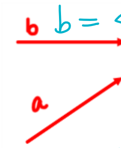
c) $\left|\frac{1}{2}\mathbf{u}\right|$

$\frac{1}{2}u = \langle \frac{1}{2}, \frac{3}{2} \rangle$

$\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2}$

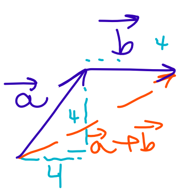
$= \sqrt{\frac{1}{4} + \frac{9}{4}} = \sqrt{\frac{10}{4}} = \frac{\sqrt{10}}{2}$

Example 7 Sketch the indicated vector using the vectors to the right.



e g $a = \langle 4, 4 \rangle$

a) $\mathbf{a} + \mathbf{b}$

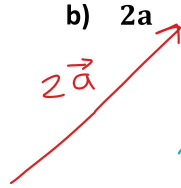


mathematically

$\vec{a} + \vec{b} = \langle 4+4, 4+0 \rangle$

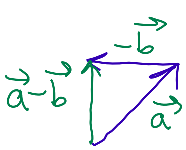
$= \langle 8, 4 \rangle$

b) $2\mathbf{a}$



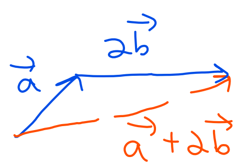
$2\vec{a} = \langle 8, 8 \rangle$

c) $\mathbf{a} - \mathbf{b}$ $\langle 4,4 \rangle - \langle 4,0 \rangle$



$\langle 0, 4 \rangle$

d) $\mathbf{a} + 2\mathbf{b}$



Multiplying vectors is not intuitive. It is similar to multiplying matrices. There are two ways to multiply vectors. We will learn one now called the dot product. The second one is called the cross product, which you will learn and use in Calculus III. Knowing how to calculate the dot product will allow us to find the angle between vectors.

Dot Product

The dot product of vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$, denoted $\mathbf{u} \cdot \mathbf{v}$ is

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta = u_1 v_1 + u_2 v_2$$

$\theta \to$ angle between $\vec{u}$ & $\vec{v}$

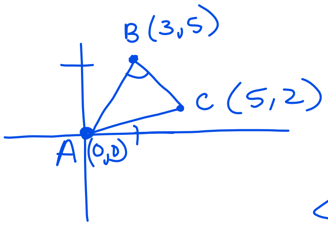
Angle Between Vectors

If θ is the angle between the nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, then

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} \text{ and } \theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right)$$

Note: If two vectors are perpendicular, their dot product is 0.

Example 8 Find the measure of $\angle ABC$ determined by the vertices $A(0,0)$, $B(3,5)$, and $C(5,2)$.



$u\ v = |u||v|\cos\theta$

2 vectors? (u) $\overrightarrow{BA}$ $\overrightarrow{BC}$ (v)

$\langle -3, -5\rangle$ $\langle 2, -3\rangle$

① $u\ v = -3\ 2 + -5\ -3 = -6 + 15 = 9$

② $|u||v| = \sqrt{9+25}\ \sqrt{4+9} = \sqrt{34}\ \sqrt{13}$

③ $\cos\theta = \dfrac{9}{\sqrt{34}\sqrt{13}}$

④ $\theta = \cos^{-1}\left(\dfrac{9}{\sqrt{34}\sqrt{13}}\right)$

⑤ $\theta = 64.653^\circ$

Example 9 Find the unit vectors tangent and normal to the curve defined by the parametric equations $\left(\frac{t}{2}+1, \sqrt{t}+1\right)$, at the point where $t = 4$.

$\frac{d}{dt}$

$x'(t) = \frac{1}{2}$

$y'(t) = \frac{1}{2}t^{-\frac{1}{2}}$ or $\frac{1}{2\sqrt{t}}$

$\left\langle \frac{1}{2}, \frac{1}{2\sqrt{t}} \right\rangle$

$\left\langle \frac{1}{2}, \frac{1}{2\sqrt{4}} \right\rangle$

$\left\langle \frac{1}{2}, \frac{1}{4} \right\rangle$

$\sqrt{\frac{1}{4}+\frac{1}{16}}$

$\sqrt{\frac{4}{16}+\frac{1}{16}}$

$\sqrt{\frac{5}{16}}$ or $\frac{\sqrt{5}}{4}$

unit vector $\left\langle \frac{1/2}{\sqrt{5}/4}, \frac{1/4}{\sqrt{5}/4} \right\rangle$

unit vector tangent $\left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$ or $\left\langle \frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \right\rangle$

unit normal $\left\langle \frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle$ or $\left\langle \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$

Example 10 Find the unit vectors tangent and normal to the curve defined by the parametric equations $(\sqrt{t}+1,\ t+1+2\sqrt{t})$ at the point where $t = 1$.

$x'(t) = \frac{1}{2}t^{-\frac{1}{2}}$
$= \frac{1}{2\sqrt{t}}$

$y'(t) = 1 + 2\cdot\frac{1}{2}t^{-\frac{1}{2}}$
$= 1 + \frac{1}{\sqrt{t}}$

Tangent $\left\langle \frac{1}{2\sqrt{t}}, 1 + \frac{1}{\sqrt{t}} \right\rangle$

Tangent at $t = 1$ $= \left\langle \frac{1}{2}, 2 \right\rangle$

mag $\sqrt{\frac{1}{4}+4}$

$\sqrt{\frac{1}{4}+\frac{16}{4}}$

$\sqrt{\frac{17}{4}} \rightarrow \frac{\sqrt{17}}{2}$

unit tangent

$\left\langle \frac{\frac{1}{2}}{\frac{\sqrt{17}}{2}}, \frac{2}{\frac{\sqrt{17}}{2}} \right\rangle$

$\left\langle \frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}} \right\rangle$

or $\left\langle -\frac{1}{\sqrt{17}}, -\frac{4}{\sqrt{17}} \right\rangle$

unit Normal

$\left\langle \frac{-4}{\sqrt{17}}, \frac{1}{\sqrt{17}} \right\rangle$

or

$\left\langle \frac{4}{\sqrt{17}}, \frac{-1}{\sqrt{17}} \right\rangle$

Example 11 A Boeing 727 airplane, flying due east at 500 mph in still air, encounters a 20 mph tailwind acting in the direction 60˚north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What are they?

$\vec{a} = \langle 500, 0 \rangle$

$\vec{w} = \langle 20\cos 60^\circ, 20\sin 60^\circ \rangle$

$\langle 10, 10\sqrt{3} \rangle$

ground velocity $= \langle 510, 10\sqrt{3} \rangle$

Speed $\sqrt{510^2 + (10\sqrt{3})^2} = \boxed{510.294 \text{ mph}}$

direction $\tan^{-1}\left(\frac{10\sqrt{3}}{510}\right)$

$\boxed{\theta = 1.945^\circ \text{ N of E}}$