BC Calculus Logistic Growth Notesheet **Name:** ______________________________

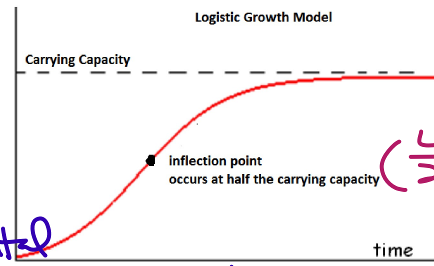
Logistic Growth

If

$$\boxed{\frac{dy}{dt} = ky(L - y)}$$

Then

$$\boxed{y = \frac{L}{1 + Ce^{-Lkt}}}$$



L

$\left(\frac{L}{2}\right)$

L is carrying capacity → the horizontal asymptote → what you're approaches

To help you remember, think "Lice Minus Licked"

—

Notice the prominence of the carrying capacity value L in each form of the equation. This is very important, especially when asked for the limit as t approaches infinity or when asked to find the y-value when the y-values are increasing most rapidly, i.e. the inflection value, which occurs at half the carrying capacity $\left(y = \frac{L}{2}\right)$.

Example 1 The population of Alaska from 1900 to 2000 can be modeled by the following logistic equation.

$$P(t) = \frac{\overset{L}{895598}}{\underset{1 + ce^{-Lkt}}{1 + 71.57e^{-0.0516t}}}$$

$.0516 = LK$

$.0516 = 895598\,K$

$K = \frac{.0516}{895{,}598}$

Where P is the population t years after 1900 with $t = 0$ corresponding to the year 1900.

a) What is the predicted population of Alaska in 2020?

$P(120) = 781{,}217.6016$

b) How fast was the population of Alaska changing in 1920? 1940? 1999?

$\frac{dy}{dt} = ky(L-y)$ / $\frac{dy}{dt}(t=20) =$ tedious $\left.\frac{dP}{dt}\right|_{t=20} = 1678.078$ | $\left.\frac{dP}{dt}\right|_{t=40} = 4127.795$

c) When was the population of Alaska growing the fastest? What was the population then? $\left.\frac{dP}{dt}\right|_{t=99} = 9741.748$

at $t = 82.765$ $P(82.765) = 447{,}798.55$

d) What information does the equation tell us about the population of Alaska in the long run?

approaches $L = (895{,}598)$

e) When $P < 447799$, What is the sign of $\frac{dP}{dt}$ and $\frac{d^2P}{dt^2}$? What does this mean in the context of the problem?

$\left(\frac{L}{2}\right)$ $\frac{dP}{dt} > 0$ $\frac{d^2P}{dt^2} > 0$ the population is growing at increasing rate

f) When $P > 447799$, What is the sign of $\frac{dP}{dt}$ and $\frac{d^2P}{dt^2}$? What does this mean in the context of the problem?

$\left(\frac{L}{2}\right)$ $\frac{dP}{dt} > 0$ $\frac{d^2P}{dt^2} < 0$ the population is growing at a decreasing rate

On the AP exam, you may be given questions that require you to recognize the parameters of logistic growth for either equation or the differential equation written in a different format. This requires you to manipulate the equation to fit one of the two standard forms.

$\left[1 - \frac{P}{100}\right] \times 100$

$100 - P$

Example 2 The growth rate of a population P of bears in a newly established wildlife preserve is modeled by the differential equation $\frac{dP}{dt} = 0.8P\left(1 - \frac{P}{100}\right)$, where t is measured in years. $\frac{dy}{dt} = ky(L-y)$

a) What is the carrying capacity for bears in this wildlife preserve?

$\frac{dP}{dt} = \frac{1}{100}(8)P(100-P)$ so $L=100$

b) What is the bear population when $\frac{dP}{dt}$ has a maximum value? What does this mean in the context of the problem?

$\frac{dP}{dt}$ maxes when $\frac{L}{2}$ = population so when $\boxed{\text{population} = 50}$

c) What is the rate of change of the population when it is growing the fastest?

$\left.\frac{dP}{dt}\right|_{P=50} = \frac{1}{100}(8)(50)(100-50) \rightarrow \frac{2500}{100}\,\frac{4}{5} \rightarrow \boxed{20 \text{ bears/yr}}$

Example 3 Suppose that a population develops according to the logistic differential equation $\frac{dP}{dt} = 0.2P - 0.002P^2$, where t is measured in weeks, and $t \geq 0$. $\frac{dP}{dt} = 002P(100 - P)$

a) If $P(0) = 5$, what is the $\lim_{t\to\infty} P(t)$?

b) If $P(0) = 60$, what is the $\lim_{t\to\infty} P(t)$? >100

c) If $P(0) = 120$, what is the $\lim_{t\to\infty} P(t)$?

d) Sketch the solution curves for a), b), and c). Which one has an inflection point? Why?

Desmos ↳ a) in $t>0$ domain (b/c initial condition is less than $\frac{L}{2}$)

Example 4 The rate at which the flu spreads through a community is modeled by the logistic differential equation $\frac{dP}{dt} = 0.001P(3000 - P)$, where t is measured in days, $t \geq 0$.

a) If $P(0) = 50$, solve for P as a function of t.

$P(t) = \frac{L}{1+ce^{-Lkt}}$

$50 = \frac{3000}{1+ce^{-3000(.001)0}}$

$50 = \frac{3000}{1+c}$

$50+50c = 3000$

$50c = 2950$

$c = 59$

$\boxed{P(t) = \frac{3000}{1+59e^{-3000\,.001t}}}$

b) Use your solution from a) to find the size of the infected population when $t = 2$ days.

$P(2) = 2617\,238$

c) Use your solution from a) to find the number of days that have occurred when the flu is spreading the fastest.

$\frac{L}{2} = 1500$ $\quad 1500 = \frac{3000}{1+59e^{-3000\,.001t}}$

$\boxed{t = 1359 \text{ days}}$