BC Calculus Logistic Growth Notesheet **Name:** ______________________________

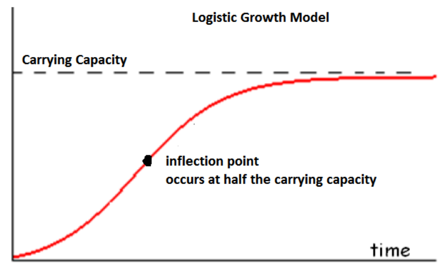
Logistic Growth

If

$$\frac{dy}{dt} = ky(L-y)$$

Then

$$y = \frac{L}{1 + Ce^{-Lkt}}$$



To help you remember, think "Lice Minus Licked"

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Notice the prominence of the carrying capacity value L in each form of the equation. This is very important, especially when asked for the limit as t approaches infinity or when asked to find the y-value when the y-values are increasing most rapidly, i.e. the inflection value, which occurs at half the carrying capacity $\left(y = \frac{L}{2}\right)$.

Example 1 The population of Alaska from 1900 to ~~2000~~ can be modeled by the following logistic equation.

$$y \quad P(t) = \frac{895598}{1 + 71.57e^{-0.0516t}} \qquad y = \frac{L}{1 + ce^{-Lkt}}$$

Where P is the population t years after 1900 with $t = 0$ corresponding to the year 1900.

units? people/yr

a) What is the predicted population of Alaska in 2020? $P(120)$

$t = 120$ ($X = 120$ in graphing calculator) $= 781{,}218$

b) How fast was the population of Alaska changing in 1920? 1940? 1999? ?

$\left.\frac{dy}{dt}\right|_{t=20} = 1{,}678078$

$\frac{dy}{dt}$ > math8 -> $\frac{d}{d\boxed{X}}[Y_1]\Big|\boxed{X} = \boxed{20}$ $\boxed{40}$. $\boxed{99}$

$\left.\frac{dy}{dt}\right|_{t=40} = 4{,}127795$

c) When was the population of Alaska growing the fastest? What was the population then?

$t \approx 82765$ (1982) $\boxed{P(82765) = 447799}$

$\left.\frac{dy}{dt}\right|_{t=99} = 9{,}741748$

d) What information does the equation tell us about the population of Alaska in the long run?

approaches 895598 $\rightarrow$ carrying capacity (End Behavior) $t \to \infty$

e) When $P < 447799$, What is the sign of $\frac{dP}{dt}$ and $\frac{d^2P}{dt^2}$? What does this mean in the context of the problem?

$\frac{dP}{dt} > 0$ $\frac{d^2P}{dt^2} > 0$ P is increasing at an increasing rate

f) When $P > 447799$, What is the sign of $\frac{dP}{dt}$ and $\frac{d^2P}{dt^2}$? What does this mean in the context of the problem?

$\frac{dP}{dt} > 0$ $\frac{d^2P}{dt^2} < 0$ P is increasing at a decreasing rate

On the AP exam, you may be given questions that require you to recognize the parameters of logistic growth for either equation or the differential equation written in a different format. This requires you to manipulate the equation to fit one of the two standard forms.

Example 2 The growth rate of a population P of bears in a newly established wildlife preserve is modeled by the differential equation $\frac{dP}{dt} = 0.8P\left(1 - \frac{P}{100}\right)$, where t is measured in years. $\frac{8}{100}\frac{8}{10}P(100-P)$

$\frac{dy}{dt} = ky(L-y)$

$\frac{dP}{dt} = \frac{8}{1000}P(100-P)$

a) What is the carrying capacity for bears in this wildlife preserve?

$\frac{dP}{dt} = \frac{8}{1000}P(100-P)$ $L = 100$

b) What is the bear population when $\frac{dP}{dt}$ has a maximum value? What does this mean in the context of the problem?

$\left(\frac{L}{2}\right) \to 50$ rate of growth is maximized

c) What is the rate of change of the population when it is growing the fastest?

$\left.\frac{dP}{dt}\right|_{P=50} = \frac{8}{1000}(50)(100-50) = \frac{400}{1000}(50) = \frac{4}{10}(50) = \frac{20}{1} = 20$ bears/yr

Example 3 Suppose that a population develops according to the logistic differential equation $\boxed{\frac{dP}{dt} = 0.2P - 0.002P^2}$, where t is measured in weeks, and $t \geq 0$.

$\frac{dP}{dt} = kP(L-P)$

$\frac{dP}{dt} = 002P(100-P)$

a) If $P(0) = 5$, what is the $\lim_{t\to\infty} P(t)$?

initial condition

$\hookrightarrow = 100$

b) If $P(0) = 60$, what is the $\lim_{t\to\infty} P(t)$?

$\hookrightarrow = 100$

c) If $P(0) = 120$, what is the $\lim_{t\to\infty} P(t)$?

$\hookrightarrow = 100$

d) Sketch the solution curves for a), b), and c). Which one has an inflection point? Why?

$\hookrightarrow$ a has a point of inflection since $\frac{L}{2} > P(0)$

Example 4 The rate at which the flu spreads through a community is modeled by the logistic differential equation $\frac{dP}{dt} = 0.001P(3000 - P)$, where t is measured in days, $t \geq 0$.

a) If $P(0) = 50$, solve for P as a function of t.

$P = \frac{3000}{1+ce^{-3000(.001)t}}$; $50 = \frac{3000}{1+ce^{-3000(.001)(0)}}$

$50 = \frac{3000}{1+c}$

$50 + 50c = 3000$

$\frac{50c}{50} = \frac{2950}{50}$

$c = 59$

$\boxed{P = \frac{3000}{1+59e^{-3t}}}$

b) Use your solution from a) to find the size of the infected population when $t = 2$ days.

$P(2) = \frac{3000}{1+59e^{-3(2)}} = 2,617\ 238$

c) Use your solution from a) to find the number of days that have occurred when the flu is spreading the fastest.

$\frac{L}{2} = \frac{3000}{2} = 1500$; graph $P = y_1 \to$ graph $y_2 = 1500$ (calculate intersection) $t = 1.359$ days