BCCALC Parametric and Vector Functions Review Solutions

1. Given $x = t^2 + 5$ and $y = e^{2t}$ find

a) $\frac{dy}{dx}$

$x' = 2t \quad y' = 2e^{2t}$

$\frac{dy}{dx} = \frac{2e^{2t}}{2t} = \boxed{\frac{e^{2t}}{t}}$

b) $\frac{d^2y}{dx^2} = \dfrac{\dfrac{t \cdot 2e^{2t} - e^{2t}(1)}{t^2}}{2t}$

$\frac{d^2y}{dx^2} = \boxed{\frac{2te^{2t} - e^{2t}}{2t^3}}$

2. Find all points at which the tangent line to the curve defined by $x = 2 - t^2$ and $y = t^3 - 4t$ are

$\frac{dy}{dx} = \frac{3t^2 - 4}{-2t}$

a) Horizontal

$3t^2 - 4 = 0$

$3t^2 = 4$

$t^2 = \frac{4}{3}$

$t = \pm \frac{2}{\sqrt{3}}$

$x\left(\frac{2}{\sqrt{3}}\right) \approx .667$

$y\left(\frac{2}{\sqrt{3}}\right) \approx -3.079$

$\boxed{(.667, -3.079)}$

$x\left(\frac{-2}{\sqrt{3}}\right) \approx .667 \quad y\left(\frac{-2}{\sqrt{3}}\right) \approx 3.079$

$\boxed{(.667, 3.079)}$

b) Vertical

$-2t = 0$

$t = 0$

$x(0) = 2 \quad y(0) = 0$

$\boxed{(2, 0)}$

3. Find the length of the curve defined by $x = 8\cos t$ and $y = 8t\sin t$, $0 \le t \le \pi/2$.

$$L = \int_0^{\frac{\pi}{2}} \sqrt{(-8\sin t)^2 + (8t\cos t + 8\sin t)^2}\,dt \approx \boxed{14.950}$$

4. If $u = \langle 2, -1 \rangle$ and $v = \langle -5, 7 \rangle$, find the following.

a) $3u + v$

$\langle 6, -3 \rangle + \langle -5, 7 \rangle$

$\boxed{\langle 1, 4 \rangle}$

b) $|3u + v|$

$\sqrt{1^2 + 4^2} = \boxed{\sqrt{17}}$

c) $u \cdot v$

$\langle 2,-1\rangle \cdot \langle -5,7\rangle$

$-10 + -7 = \boxed{-17}$

d) The angle between u and v.

$\theta \approx \cos^{-1}\left(\frac{-17}{\sqrt{5}\cdot\sqrt{74}}\right)$

$\boxed{\theta \approx 152.103^\circ}$

5. Find the unit vectors that are tangent and normal to the curve defined by $x = 2t^3 - 3t$ and $y = -5t^2$ at $t = 1$.

$x' = 6t^2 - 3$ $\quad y' = -10t$

$\langle 3, -10\rangle$ $\quad \sqrt{3^2 + (-10)^2} = \sqrt{109}$

$\frac{1}{\sqrt{109}}\langle 3, -10\rangle = \left\langle \frac{3}{\sqrt{109}}, \frac{-10}{\sqrt{109}} \right\rangle$

UNIT TAN

$\left\langle \frac{3}{\sqrt{109}}, \frac{-10}{\sqrt{109}} \right\rangle \left\langle \frac{-3}{\sqrt{109}}, \frac{10}{\sqrt{109}} \right\rangle$

UNIT NORMAL

$\left\langle \frac{10}{\sqrt{109}}, \frac{3}{\sqrt{109}} \right\rangle \left\langle \frac{-10}{\sqrt{109}}, \frac{-3}{\sqrt{109}} \right\rangle$

6. Given $r(t) = (\sec t)\mathbf{i} + (\tan t)\mathbf{j}$ models the position of an object, find the following.

a) Find the velocity vector at time $t = \frac{\pi}{6}$.

$r'(t) = \sec t \tan t \, i + \sec^2 t \, j$

$r'\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} i + \left(\frac{2}{\sqrt{3}}\right)^2$

$\boxed{v\left(\frac{\pi}{6}\right) = \frac{2}{3} i + \frac{4}{3} j}$

6. Given $r(t) = (\sec t)\mathbf{i} + (\tan t)\mathbf{j}$ models the position of an object, find the following.

b) Find the acceleration vector.

$r'(t) = \sec t \tan t \, i + \sec^2 t \, j$

$r''(t) = (\sec t \cdot \sec^2 t + \tan t \cdot \sec t \cdot \tan t) i + 2 \sec t \cdot \sec t \cdot \tan t \, j$

$\boxed{a(t) = (\sec^3 t + \sec t \tan^2 t) i + 2\sec^2 t \tan t \, j}$

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6. Given $r(t) = (\sec t)\mathbf{i} + (\tan t)\mathbf{j}$ models the position of an object, find the following.

c) Find the speed of the object at time $t = \frac{\pi}{6}$.

$$\sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{4}{3}\right)^2} = \sqrt{\frac{4}{9} + \frac{16}{9}} = \boxed{\frac{\sqrt{20}}{3}}$$

d) Find a unit vector in the direction of the object's velocity at time $t = \frac{\pi}{6}$. No decimal answers.

$$\frac{3}{\sqrt{20}}\left\langle \frac{2}{3}, \frac{4}{3} \right\rangle = \boxed{\left\langle \frac{2}{\sqrt{20}}, \frac{4}{\sqrt{20}} \right\rangle}$$

7. Given $r(t) = (\sec t)\mathbf{i} + (\tan t)\mathbf{j}$ models the position of an object, find the following.

a) Write an equation for the line tangent to the path of the object at the point where $t = -1$.

$r'(t) = \sec t \tan t\, i + \sec^2 t\, j$ $\quad r(-1) \approx 1.851 i + {}^{-}1.577 j \rightarrow (1.851, -1.577)$

$\frac{dy}{dx} \approx -1.188$ $\quad \boxed{y + 1.577 = -1.188(x - 1.851)}$

b) Write an equation for the line normal to the path of the object at the point where $t = -1$.

$m\perp = \frac{-1}{-1.189} \approx .841$ $\quad \boxed{y + 1.577 = .841(x - 1.851)}$

8. Evaluate the integral $\int \left((6 - 6t)\mathbf{i} + (3\sqrt{t})\mathbf{j}\right) dt$

$(6t - 3t^2)i + (2t^{\frac{3}{2}})j + C$

9. Find a vector function for $\mathbf{r}(t)$ if $\frac{dr}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$ and $r(0) = \mathbf{j}$

$r(t) = \cos t\, i + \sin t\, j + C$

$j = i + c \qquad c = -i + j$

$$\boxed{r(t) = (\cos t - 1)i + (\sin t + 1)j}$$

10. At time t, $0 \le t \le 4$, the position of a particle moving along a path in a plane is given by the parametric equations $x(t) = e^t \cos t$ and $y(t) = e^t \sin t$.

a) Find the slope of the path of the particle when $t = \pi$.

$y'(\pi) \approx -23.141$

$x'(\pi) \approx -23.141$

$$\boxed{\frac{dy}{dx} \approx 1}$$

10. At time t, $0 \leq t \leq 4$, the position of a particle moving along a path in a plane is given by the parametric equations $x(t) = e^t \cos t$ and $y(t) = e^t \sin t$.

b) Find the speed of the particle when $t = 3$.

$$\sqrt{(x'(3))^2 + (y'(3))^2} \approx \boxed{28.405}$$

10. At time t, $0 \leq t \leq 4$, the position of a particle moving along a path in a plane is given by the parametric equations $x(t) = e^t \cos t$ and $y(t) = e^t \sin t$.

c) Find the distance traveled by the particle along the path from $t = 0$ to $t = 3$.

$$\int_0^3 \sqrt{(-e^t \sin t + e^t \cos t)^2 + (e^t \cos t + e^t \sin t)^2} \approx \boxed{26.991}$$

10. At time t, $0 \le t \le 4$, the position of a particle moving along a path in a plane is given by the parametric equations $x(t) = e^t \cos t$ and $y(t) = e^t \sin t$.

d) Is the vertical movement of the particle up or down at time $t = 3$? Give a reason for your answer.

$y'(3) \approx -17.050$

down since $y'(3) < 0$

11. An object moving along a curve in the xy-plane has position $(x(t), y(t))$ with $\frac{dx}{dt} = \cos(t^2)$ and $\frac{dy}{dt} = \sin(t^3)$. At time $t = 3$, the object is at position $(4, 7)$. Find the position of the particle at time $t = 1$.

$x(1) = 4 + \int_3^1 \cos(t^2)\,dt$

4.202

$y(1) = 7 + \int_3^1 \sin(t^3)\,dt$

6.777

$\boxed{(4.202, 6.777)}$

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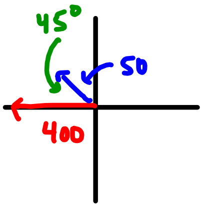
12. An airplane is headed due West at a speed of 400 mph and encounters a 50 mph wind blowing in the northwest direction (45° N of W). The airplane maintains its compass heading but, because of the wind, acquires a new ground speed and direction. Find the new ground speed in mph and the new direction in degrees of the plane. Make sure to change your calculator to DEGREE mode and change it back to radian mode when you are finished.



$$\text{plane} = \langle -400, 0 \rangle$$

$$\text{wind} = \langle 50\cos 135°, 50\sin 135° \rangle = \langle -25\sqrt{2}, 25\sqrt{2} \rangle$$

$$\text{plane} + \text{wind} = \langle -400 - 25\sqrt{2}, 25\sqrt{2} \rangle$$

$$|\text{plane} + \text{wind}| = \boxed{436.789 \text{ mph}}$$

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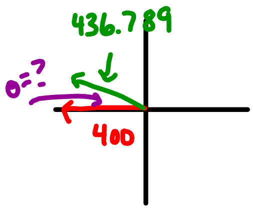

$$\text{plane} = \langle -400, 0 \rangle$$

$$\text{plane} + \text{wind} = \langle -400 - 25\sqrt{2}, 25\sqrt{2} \rangle$$

$$|\text{plane} + \text{wind}| = 436.789 \text{ mph}$$

$$\theta = \cos^{-1}\left(\frac{-400(-400-25\sqrt{2}) + 0}{400 \cdot 436.789} \right)$$

$$\boxed{\theta \approx 4.643° \text{ N of W}}$$