Example 5 Given $x = 2\sqrt{t}$, $y = 3t^2 - 2t$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and evaluate $t = 1$.

$\frac{dx}{dt} = 2\frac{1}{2\sqrt{t}} = \frac{1}{\sqrt{t}}$ or $t^{-\frac{1}{2}}$

$\frac{dy}{dt} = 6t - 2$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6t-2}{t^{-\frac{1}{2}}}$

$\boxed{\frac{dy}{dx} = 6t^{\frac{3}{2}} - 2t^{\frac{1}{2}}}$

$\boxed{\left.\frac{dy}{dx}\right|_{t=1} = 6 - 2 = 4}$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \rightarrow \frac{d}{dt}\left(6t^{\frac{3}{2}} - 2t^{\frac{1}{2}}\right) = \frac{9t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{t^{-\frac{1}{2}}}$

$\boxed{\frac{d^2y}{dx^2} = 9t - 1}$

$\boxed{\left.\frac{d^2y}{dx^2}\right|_{t=1} = 9 - 1 = 8}$

Example 6 Given $x = 4\cos t$, $y = 3\sin t$, write an equation of the tangent line to the curve at the point where $t = \frac{3\pi}{4}$.

$\boxed{x, y, \frac{dy}{dx}}$

$x\left(\frac{3\pi}{4}\right) \Rightarrow x = 4\cos\left(\frac{3\pi}{4}\right) = 4\left(\frac{-\sqrt{2}}{2}\right) = -2\sqrt{2}$

$y\left(\frac{3\pi}{4}\right) = 3\sin\left(\frac{3\pi}{4}\right) = \frac{3\sqrt{2}}{2}$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$\frac{dx}{dt} = -4\sin t$

$\frac{dy}{dt} = 3\cos t$

$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3\cos t}{-4\sin t} \Rightarrow$ at $t = 3/4$ $= \frac{3\left(-\frac{\sqrt{2}}{2}\right)}{-4\left(\frac{\sqrt{2}}{2}\right)} = \frac{3}{4}$

$\boxed{y - \frac{3\sqrt{2}}{2} = \frac{3}{4}\left(x + 2\sqrt{2}\right)}$

Example 7 Find all points of horizontal and vertical tangency given $x = t^2 + t$ and $y = t^2 - 3t + 5$.

H.T $\frac{dy}{dx} = 0$

$\frac{dy}{dt} = 2t - 3$

$0 = 2t - 3$ $\left(t = \frac{3}{2}\right)$

V T $\frac{dy}{dx}$ is undefined $\left(\frac{\text{Not } 0}{0}\right)$

$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$

$\frac{dx}{dt} = 2t + 1$

$0 = 2t + 1$ $\left(t = -\frac{1}{2}\right)$

Point of V T

$x\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^2 - \frac{1}{2} = \frac{1}{4} - \frac{2}{4} = -\frac{1}{4}$

$y\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^2 - 3\left(-\frac{1}{2}\right) + 5 = \frac{1}{4} + \frac{6}{4} + \frac{20}{4} = \frac{27}{4} \rightarrow \left(-\frac{1}{4}, \frac{27}{4}\right)$

Point of H T $(t = 3/2)$ $\left(\frac{15}{4}, \frac{11}{4}\right)$

$x\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 + \frac{3}{2} \rightarrow \frac{9}{4} + \frac{6}{4} = \frac{15}{4}$

$y\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) + 5 \rightarrow \frac{9}{4} - \frac{18}{4} + \frac{20}{4} = \frac{11}{4}$

Parametric Arc Length

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Is the length of the arc from $t = a$ to $t = b$.

$\frac{dx}{dt} \rightarrow$ Velocity in x direction

$\frac{dy}{dt} \rightarrow$ velocity in y direction

Example 8 Find the arc length of the given curve if $x = t^2$, $y = 4t^3 - 1$, $0 \le t \le 1$.

$\frac{dx}{dt} = 2t$

$\frac{dy}{dt} = 12t^2$

$L = \int_0^1 \sqrt{(2t)^2 + (12t^2)^2}\, dt$

≈ 4.149

Particle Motion with Parametric Equations

$x'(t) = \frac{dx}{dt}$	the rate at which the x-coordinate is changing with respect to t or the velocity of an object in the horizontal direction.
$y'(t) = \frac{dy}{dt}$	the rate at which the y-coordinate is changing with respect to t or the velocity of an object in the vertical direction.
$(x(t), y(t))$	the position at any time t.
$(x'(t), y'(t))$	the velocity at any time t.
$(x''(t), y''(t))$	the acceleration at any time t.
$\frac{dy}{dx}$	the rate of change of y with respect to x or the slope of the tangent line to the curve.
$\frac{d^2y}{dx^2}$	the rate of change of the slope of the curve with respect to x.
$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$	The speed of a particle
$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$	the distance traveled by a particle from $t = a$ to $t = b$.

Example 9 A particle moves in the xy-plane so that any time t, $t \geq 0$, the position of the particle is given by $x(t) = t^3 + 4t^2$, $y(t) = t^4 - t^3$.

a) Find the parametric equations representing the velocity of the particle. Then find the horizontal and vertical components of the velocity at $t = 1$.

$x'(t) = 3t^2 + 8t \quad y'(t) = 4t^3 - 3t^2$

$x'(1) = 3(1)^2 + 8(1) = 11$

$y'(1) = 4(1)^3 - 3(1)^2 = 1$

b) Find the speed of the particle at $t = 1$.

$S = \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}$

speed at $t=1$ $= \sqrt{1^2 + 11^2} = \sqrt{122} \approx 11045$

c) Find the parametric equations representing the acceleration of the particle. Then find the horizontal and vertical components of the acceleration at $t = 1$.

$x''(t) = 6t + 8 \quad y''(t) = 12t^2 - 6t$

$x''(1) = 6(1) + 8 = 14$

$y''(1) = 12(1)^2 - 6(1) = 6$

d) Find the distance the particle travels from time $t = 2$ to $t = 5$.

$\int_2^5 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt \approx 536.462$