BC Calculus Partial Fractions Notesheet **Name:** ______________________________

To begin: Add then simplify the following rational expressions by finding a common denominator.

$$\frac{(x-3)}{(x-3)}\cdot\frac{5}{2x-1}-\frac{2}{(x-3)}\cdot\frac{(2x-1)}{(2x-1)}$$

$$\frac{5x-15-4x+2}{(x-3)(2x-1)}=\frac{x-13}{(x-3)(2x-1)}$$

Evaluate $\int \frac{x-13}{2x^2-7x+3}dx$

$$\frac{x-13}{(2x-1)(x-3)}=\frac{A(x-3)}{(2x-1)(x-3)}+\frac{B(2x-1)}{(x-3)(2x-1)}$$

$x-13= A(x-3)+B(2x-1)$

if $x=3$ $\quad 3-13=A(0)+B(5) \rightarrow -10=5B \rightarrow B=-2$

if $x=\frac{1}{2}$ $\quad \frac{1}{2}-13=A(\frac{1}{2}-3)+B(0) \rightarrow -\frac{25}{2}=-\frac{5}{2}A \rightarrow 5$

$$\int\left(\frac{5}{2x-1}+\frac{-2}{x-3}\right)dx \rightarrow \frac{5}{2}\ln|2x-1|-2\ln|x-3|+c$$

The previous example was made easier because we already knew an equivalent form of the rational expression we needed to integrate. The question becomes, how do you find the equivalent form when it is not given to you? The answer is the Partial Fraction Decomposition method. The Partial Fraction Decomposition technique is another algebraic massage technique that can be used to transform an expression into one that allows us to integrate. In this class, you will be expected to decompose rational expressions involving non-repeating, linear factors in the denominator. For cases like this one, there is a very slick method you can use that was developed by Oliver Heaviside, called the Heaviside Cover-Up method.

Example 1 $\int \frac{x+5}{x^2+x-2}dx$

$x^2+x-2=(x+2)(x-1)$

$$\frac{x+5}{x^2+x-2}=\frac{A(x-1)}{(x+2)(x-1)}+\frac{B(x+2)}{(x-1)(x+2)}$$

$x+5=A(x-1)+B(x+2)$

if $x=1$ $\quad 6=A(0)+3B \quad B=2$

if $x=-2$ $\quad 3=A(-3)+B(0) \quad A=-1$

$$\int\left(\frac{-1}{x+2}+\frac{2}{x-1}\right)dx \rightarrow -\ln|x+2|+2\ln|x-1|+c$$

Example 2 $\int \frac{7x}{(2x-3)(x+2)}dx$

$$\frac{7x}{(2x-3)(x+2)}=\frac{A(x+2)}{(2x-3)(x+2)}+\frac{B(2x-3)}{(x+2)(2x-3)}$$

$7x=A(x+2)+B(2x-3)$

if $x=-2$ $\quad -14=B(-7) \quad B=2$

if $x=\frac{3}{2}$ $\quad \frac{21}{2}=\frac{7A}{2} \quad A=3$

$$\int\left(\frac{3}{2x-3}+\frac{2}{x+2}\right)dx=\frac{3}{2}\ln|2x-3|+2\ln|x+2|+c$$

chain rule

Example 3 $\int \frac{3}{(x-1)(x+2)}dx$

$$=\ln|x-1|-\ln|x+2|+c$$

or as condensed $\ln\left|\frac{x-1}{x+2}\right|+C$

$$* \quad \frac{4}{x^2-1} = \frac{A(x-1)}{(x+1)(x-1)} + \frac{B(x+1)}{(x-1)(x+1)}$$

$4 = A(x-1) + B(x+1)$

if $x=1 \to 4=2B \to B=2$

if $x=-1$ $\to 4=-2A \to A=-2$

In order for partial fraction decomposition to work, the degree of the denominator must be greater than the degree of the numerator. When it's not, we'll use our old friend Mr. Long Division first.

Example 4 $\int \frac{3x^4+1}{x^2-1}dx = \int\left(3x^2+3+\overset{*}{\frac{4}{x^2-1}}\right)dx$

$$\begin{array}{r} 3x^2+3 \\ x^2-1 \overline{\big)\, 3x^4+1} \\ -(3x^4-3x^2) \\ \hline 3x^2+1 \\ -(3x^2-3) \\ \hline 4 \end{array}$$

$$= \int\left(3x^2+3-\frac{2}{x+1}+\frac{2}{x-1}\right)dx$$

$$= x^3+3x-2\ln|x+1|+2\ln|x-1|+C$$

Introducing Logistic Growth

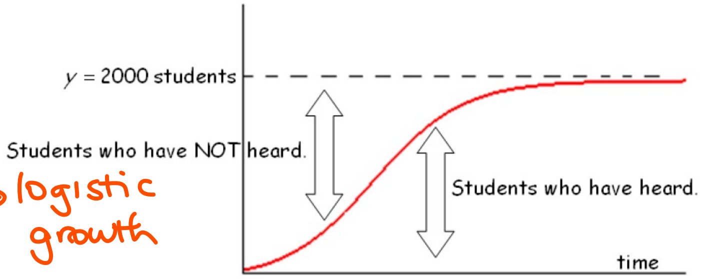
Many things that grow exponentially cannot continue to do so indefinitely. After a while, things that start off growing exponentially begin to compete for resources like food, water, money, and parking spaces. The growth begins to taper off as it approaches some **carrying capacity** of the system.

In this case, the growth rate is not only proportional to the current value ($y = Ce^{kt}$), but also how far the current value is from the carrying capacity.

Imagine a rumor spreading through a school of 2000 students. The rate at which the rumor spreads is directly proportional to both the students who have heard the rumor and the students who have yet to hear the rumor as the number of people hearing the rumor approaches 2000. The curve for the spread of the rumor might look like something shown below. This type of curve and growth is called a Logistic Growth Curve.

$$* \quad y = \frac{Lce^{Lkt}/ce^{Lkt}}{\frac{1+ce^{Lkt}}{ce^{Lkt}}\ \ \frac{}{ce^{Lkt}}}$$

$$\boxed{y = \frac{L}{1+ce^{-Lkt}}} \to \text{logistic growth}$$



For quantities, y, that grow logistically with a carrying capacity of $y = L$, we can state the relation mathematically in the following way

$$\boxed{\frac{dy}{dt} = ky(L-y)}$$

*Exponential growth $\frac{dy}{dt} = ky$

Example 5 Solve the differential equation $\frac{dy}{dt} = ky(L-y)$

$dy = ky(L-y)\,dt$

$$\int \frac{dy}{y(L-y)} = \int k\,dt$$

$$\frac{1}{y(L-y)} = \frac{A(L-y)}{y(L-y)} + \frac{B\,y}{(L-y)\,y}$$

$1 = A(L-y) + By$

if $y = L \to 1 = BL \to B = \frac{1}{L}$

if $y = 0 \to 1 = AL \to A = \frac{1}{L}$

$$\int\left(\frac{1}{Ly} + \frac{1}{L(L-y)}\right)dy$$

$$\Rightarrow \frac{1}{L}\int\left(\frac{1}{y} + \frac{1}{L-y}\right)dy = \int k\,dt$$

$\frac{1}{L}\left(\ln|y| - \ln|L-y|\right) = kt+c$

$\ln|y| - \ln|L-y| = Lkt+c$

$e^{\ln\left|\frac{y}{L-y}\right|} = e^{Lkt+c}$

$\frac{y}{L-y} = ce^{Lkt}$

$y = (L-y)ce^{Lkt}$

$y = Lce^{Lkt} - yce^{Lkt}$

$y + yce^{Lkt} = Lce^{Lkt}$

$y(1+ce^{Lkt}) = Lce^{Lkt}$ *