BCCALC Polar Arc Length Homework Solutions

1. Convert the following polar coordinates to rectangular coordinates.

a) $(-4, \pi/3)$

$x = -4\cos\frac{\pi}{3}$ $y = -4\sin\frac{\pi}{3}$

$x = -4\left(\frac{1}{2}\right)$ $y = -4\left(\frac{\sqrt{3}}{2}\right)$

$x = -2$

$\boxed{(-2, -2\sqrt{3})}$

b) $(-8, \pi)$

$x = -8\cos\pi$ $y = -8\sin\pi$

$x = 8$ $y = 0$

$\boxed{(8, 0)}$

2. Convert the following rectangular coordinates to polar coordinates.

a) $(4, 4)$ ← QI $(+,+)$

$4^2 + 4^2 = r^2$ $\tan\theta = \frac{4}{4}$

$r^2 = 32$ $\tan\theta = 1$

$r = \sqrt{32}$ $\theta = \frac{\pi}{4}$

$\boxed{\left(\sqrt{32}, \frac{\pi}{4}\right)}$

b) $(0, \sqrt{6})$ ← on pos. y-axis

$0^2 + (\sqrt{6})^2 = r^2$

$r^2 = 6$ $\tan\theta = \frac{\sqrt{6}}{0}$

$r = \sqrt{6}$ $\tan\theta = $ und

$\theta = \frac{\pi}{2}$

$\boxed{\left(\sqrt{6}, \frac{\pi}{2}\right)}$

3. Convert the following polar equations to rectangular equations.

a) $6r\cos\theta + 7r\sin\theta = 1$

$6x+7y=1$

$7y=-6x+1$

$\boxed{y=-\frac{6}{7}x+\frac{1}{7}}$

b) $r = 24\sin\theta$

$r^2 = 24r\sin\theta$

$x^2+y^2=24y$

complete square $\left(\frac{-24}{2}\right)^2 = 144$

$x^2+y^2-24y=0$

$x^2+y^2-24y+144=144$

$\boxed{x^2+(y-12)^2=144}$

4. Find the slope of the polar curve $r = 5\cos(3\theta)$ at the point where $\theta = \pi/3$.

$x=5\cos(3\theta)\cos\theta$

$x'=5\cos(3\theta)(-\sin\theta)+\cos\theta\cdot -15\sin(3\theta)$

$x'\left(\frac{\pi}{3}\right)=5\cos\pi\left(-\sin\frac{\pi}{3}\right)+\cos\frac{\pi}{3}\cdot -15\sin\pi$

$= -5\left(-\frac{\sqrt{3}}{2}\right)+0$

$= \frac{5\sqrt{3}}{2}$

$y=5\cos(3\theta)\sin\theta$

$y'=5\cos(3\theta)\cos\theta+\sin\theta\cdot -15\sin(3\theta)$

$y'\left(\frac{\pi}{3}\right)=5\cos\pi\cos\frac{\pi}{3}+\sin\frac{\pi}{3}\cdot -15\sin\pi$

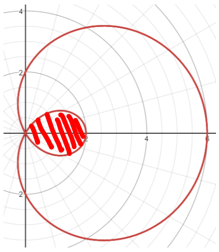
$= -5\left(\frac{1}{2}\right)+0$

$= -\frac{5}{2}$

$$\frac{dy}{dx} = \frac{\frac{-5}{2}}{\frac{5\sqrt{3}}{2}} = \frac{-5}{2}\cdot\frac{2}{5\sqrt{3}} = \boxed{\frac{-1}{\sqrt{3}}}$$

5. Find the area of the region.

a) Inside the inner loop of $r = 2 + 4\cos\theta$

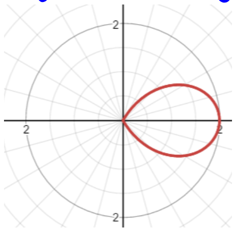


$0 = 2 + 4\cos\theta$

$-\frac{1}{2} = \cos\theta$

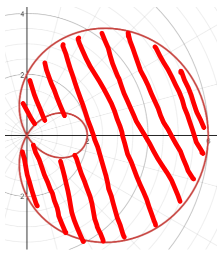
$\theta = \frac{2\pi}{3}$ $\theta = \frac{4\pi}{3}$

graph from $\frac{2\pi}{3}$ to $\frac{4\pi}{3}$



$$A = \frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (2+4\cos\theta)^2 \, d\theta \approx \boxed{2.174}$$

b) Inside the outer loop and outside the inner loop of circle $r = 2 + 4\cos\theta$



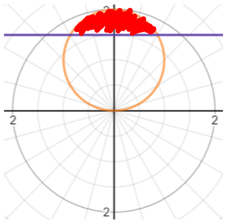
OUTER − INNER

$$\frac{1}{2}\int_{\frac{4\pi}{3}}^{\frac{8\pi}{3}} (2+4\cos\theta)^2 \, d\theta - \frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (2+4\cos\theta)^2 \, d\theta \approx \boxed{33.351}$$

Whole − 2 Loops

$$\frac{1}{2}\int_{0}^{2\pi} (2+4\cos\theta)^2 \, d\theta - 2\cdot\frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (2+4\cos\theta)^2 \, d\theta \approx \boxed{33.351}$$

c) Inside $r = 2\sin\theta$ and above the line $r = \frac{3}{2}\csc\theta$.



$2\sin\theta = \frac{3}{2\sin\theta}$

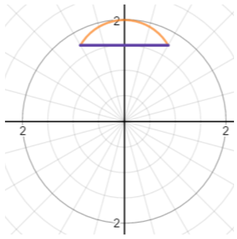
$4\sin^2\theta = 3$

$\sin^2\theta = \frac{3}{4}$

$\sin\theta = \pm\frac{\sqrt{3}}{2}$

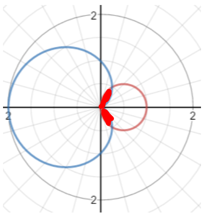
$\theta = \frac{\pi}{3}$ $\theta = \frac{2\pi}{3}$

graph from $\frac{\pi}{3}$ to $\frac{2\pi}{3}$



$$A = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \left((2\sin\theta)^2 - \left(\frac{3}{2}\csc\theta\right)^2\right) d\theta \approx \boxed{.614}$$

d) The common interior of $r = (\cos\theta)$ and $r = (1 - \cos\theta)$.



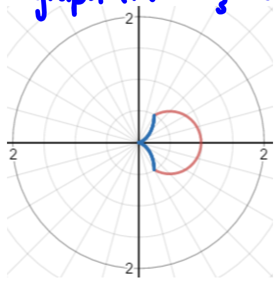
$\cos\theta = 1 - \cos\theta$

$2\cos\theta = 1$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}$ $\theta = \frac{5\pi}{3}$ $\theta = \frac{-\pi}{3}$

graph from $\frac{-\pi}{3}$ to $\frac{\pi}{3}$



Whole red circle — Part in graph above

$$\frac{1}{2}\int_0^{\pi} (\cos\theta)^2 d\theta - \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left((\cos\theta)^2 - (1-\cos\theta)^2\right) d\theta \approx \boxed{.101}$$

6. Find the length of the curve

a) The parabolic segment $r = \frac{6}{(1+cos\theta)}$, $0 \le \theta \le \frac{\pi}{2}$

$r = 6(1+\cos\theta)^{-1}$

$\frac{dr}{d\theta} = -6(1+\cos\theta)^{-2}(-\sin\theta) = \frac{6\sin\theta}{(1+\cos\theta)^2}$

$$L = \int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{6}{1+\cos\theta}\right)^2 + \left(\frac{6\sin\theta}{(1+\cos\theta)^2}\right)^2}\, d\theta \approx \boxed{6.887}$$

b) $r = 1 + \cos\theta, 0 \le \theta \le 2\pi$

$\frac{dr}{d\theta} = -\sin\theta$

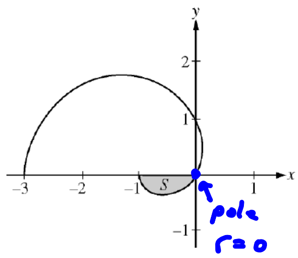
$$L = \int_0^{2\pi} \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta \approx \boxed{8}$$

c) $r = \cos^3\left(\frac{\theta}{3}\right)$, $0 \le \theta \le \frac{\pi}{4}$

$$\frac{dr}{d\theta} = 3\left(\cos\left(\frac{\theta}{3}\right)\right)^2\left(-\sin\left(\frac{\theta}{3}\right)\right)\cdot\left(\frac{1}{3}\right) = -\cos^2\left(\frac{\theta}{3}\right)\sin\left(\frac{\theta}{3}\right)$$

$$L = \int_0^{\frac{\pi}{4}} \sqrt{\left(\cos^3\left(\frac{\theta}{3}\right)\right)^2 + \left(-\cos^2\left(\frac{\theta}{3}\right)\sin\left(\frac{\theta}{3}\right)\right)^2}\, d\theta \approx \boxed{.768}$$

7. The graph of the polar curve $r = 1 - 2\cos\theta$ for $0 \le \theta \le \pi$ is shown to the right. Let S be the shaded region on the third quadrant bounded by the curve and the x-axis.



a) Write an integral expression for the area of S.

$1 - 2\cos\theta = 0$

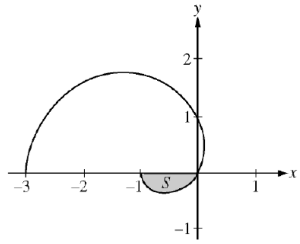
$1 = 2\cos\theta$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}$

$$\boxed{A = \frac{1}{2}\int_0^{\frac{\pi}{3}} (1 - 2\cos\theta)^2\, d\theta}$$

7. The graph of the polar curve $r = 1 - 2\cos\theta$ for $0 \le \theta \le \pi$ is shown to the right. Let S be the shaded region on the third quadrant bounded by the curve and the x-axis.



b) Write expressions for $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ in terms of θ.

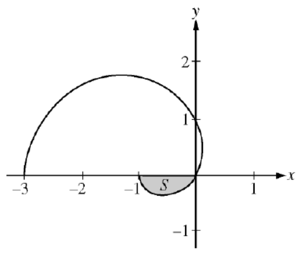
$x = \cos\theta - 2\cos^2\theta$

$y = \sin\theta - 2\cos\theta\sin\theta$

$\boxed{\frac{dx}{d\theta} = -\sin\theta - 4\cos\theta(-\sin\theta)}$

$\boxed{\frac{dy}{d\theta} = \cos\theta - 2\cos\theta\cos\theta + \sin\theta(2\sin\theta)}$

7. The graph of the polar curve $r = 1 - 2\cos\theta$ for $0 \le \theta \le \pi$ is shown to the right. Let S be the shaded region on the third quadrant bounded by the curve and the x-axis.



c) Write an equation in terms of x and y for the line tangent to the graph of the polar curve at the point where $\theta = \pi/2$. Show the computations that lead to your answer.

$\frac{dx}{d\theta} = -\sin\theta - 4\cos\theta(-\sin\theta)$

$\frac{dy}{d\theta} = \cos\theta - 2\cos\theta\cos\theta + \sin\theta(2\sin\theta)$

$\left.\frac{dx}{d\theta}\right|_{\theta=\frac{\pi}{2}} = -1$

$\left.\frac{dy}{d\theta}\right|_{\theta=\frac{\pi}{2}} = 1$

$\frac{dy}{dx} = \frac{1}{-1} = \boxed{-1}$