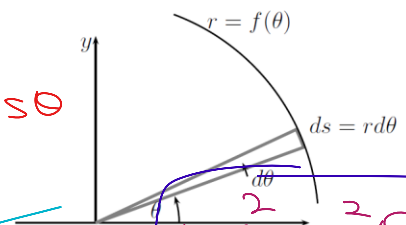
BC Calculus Polar Arc Length Notesheet Name: ____________________

In this section, we will learn how to find the length of polar curves given by

$$r = f(\theta) \text{ over the interval } \alpha \le \theta \le \beta$$

where we assume the curve is traced exactly once. Just like arc length for functions in terms of x and arc length for parametric equations, we will use a formula. However, let's see where it comes from first. Just as we did when finding slope of tangent lines in polar coordinates, we will start by writing the curve in terms of a set of parametric equations by finding an x function and a y function.

Example 1: Derive the formula for polar arc length



$x = r\cos\theta$ $\quad$ $y = r\sin\theta$

$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta + r(-\sin\theta)$ $\quad$ $\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$

$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$

$\left(\frac{dy}{d\theta}\right)^2 = \boxed{\left(\frac{dr}{d\theta}\right)^2 \sin^2\theta} + \cancel{2\frac{dr}{d\theta}\sin\theta\, r\cos\theta} \boxed{+ r^2\cos^2\theta}$

$\left(\frac{dx}{d\theta}\right)^2 = \boxed{\left(\frac{dr}{d\theta}\right)^2 \cos^2\theta} + \cancel{2\frac{dr}{d\theta}\cos\theta\, r(-\sin\theta)} \boxed{+ r^2\sin^2\theta}$

$\left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta$

$\left(\frac{dr}{d\theta}\right)^2 \left(\underbrace{\sin^2\theta + \cos^2\theta}_{1}\right)$

$r^2 + \left(\frac{dr}{d\theta}\right)^2$

Polar Arc Length

The length of a curve with polar equation $r = f(\theta)$ over the interval $\alpha \le \theta \le \beta$ is

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$a, b \in \theta$

Example 2 Find the length of the curve $r = 2 - 2\cos\theta$ over the interval $0 \le \theta \le 2\pi$.

$\frac{dr}{d\theta} = -2(-\sin\theta)$

$$L = \int_0^{2\pi} \sqrt{(2-2\cos\theta)^2 + (2\sin\theta)^2}\, d\theta = 16$$

Example 3 Find the length of the curve $r = 2\cos\theta$ over the interval $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\frac{dr}{d\theta} = -2\sin\theta$

$$L = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{(2\cos\theta)^2 + (-2\sin\theta)^2}\, d\theta \approx 6.283$$