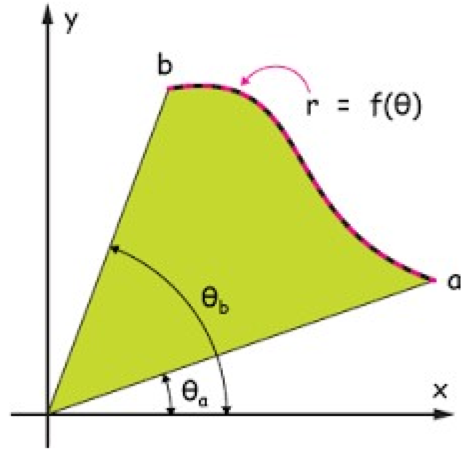
BC Calculus Polar Area Notesheet **Name:** ____________________________

We are going to look at areas enclosed by polar curves, that's enclosed, not under as we typically have in these problems. These problems work a little differently in polar coordinates. Here is a sketch of what the area that we'll be finding in this section looks like.



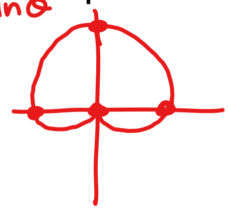
The formula for polar area is different from all previous area formulas, because it is not based on rectangles. Instead, polar area uses an infinite number of sectors to find the area. Remember that a sector is a hunk of a circle, a slice of pizza from the whole pizza.

The area of a sector of a circle is given by $A = \frac{1}{2}\theta r^2$, where θ is measured in radians. Our area is bounded by the radial lines from $\theta = \alpha$ to $\theta = \beta$.

$$Area = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$$

Example 1 Find the area bounded by the graph of $r = 2 + 2\sin\theta$.

θ	$r=2+2\sin\theta$
0	2
$\pi/2$	4
π	2
$3\pi/2$	0
2π	2



$A = \frac{1}{2}\int_0^{2\pi} (2+2\sin\theta)^2 d\theta = 18.850$

Example 2 Find the area of one petal of $r = 2\sin(3\theta)$ → odd → so 3 petals → entire rose in π rotation

$\pi/3$

$r = 0$ $\quad 0 = 2\sin(3\theta)$

$0 = \sin(3\theta)$

$0, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{3}, \frac{4\pi}{3} \ldots = 3\theta$

$A_{1\,petal} = \frac{1}{2}\int_0^{\pi/3} (2\sin 3\theta)^2 d\theta = 1.047$

Example 3 Find the area of one petal of $r = \cos(2\theta)$ ↳ even → so $\underline{\underline{4}}$ petals → entire rose in 2π rotation

$A_{1\,petal} = \frac{1}{2}\int_0^{\pi/2} (\cos 2\theta)^2 d\theta$

Example 4 Find the area inside one loop of $r^2 = 4\cos(2\theta)$

↳ Infinity symbol

$0 = 4\cos 2\theta$

$0 = \cos 2\theta$

$\frac{-\frac{\pi}{2}}{2}, \frac{\frac{\pi}{2}}{2}, \frac{\frac{3\pi}{2}}{2}, \frac{\frac{5\pi}{2}}{2}$, etc → $= \frac{2\theta}{2}$

$-\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \ldots$

θ	r^2
$-\pi/4$	0
0	4
$\pi/4$	0
$\pi/2$	-4
$3\pi/4$	0

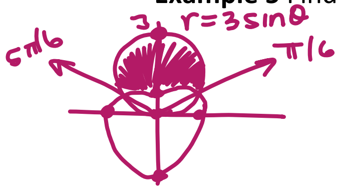
Bad

θ	
π	4
$5\pi/4$	0
$3\pi/2$	-4

$A_{1\,Loop} = \frac{1}{2}\int_{-\pi/4}^{\pi/4} 4\cos(2\theta)\, d\theta$

$\boxed{= 2}$

Example 5 Find the area inside $r = 3\sin\theta$ and outside $r = 2 - \sin\theta$

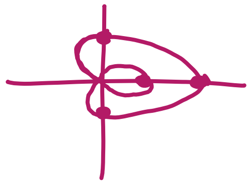


$3\sin\theta = 2 - \sin\theta$
$4\sin\theta = 2$
$\sin\theta = 1/2$
$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$$A = \frac{1}{2}\int_{\pi/6}^{5\pi/6}\left((3\sin\theta)^2 - (2-\sin\theta)^2\right)d\theta$$

Example 6 Find the area of the inner loop of the graph of $r = 1 + 2\cos\theta$

θ	r
0	3
$\pi/2$	1
π	-1
$3\pi/2$	1



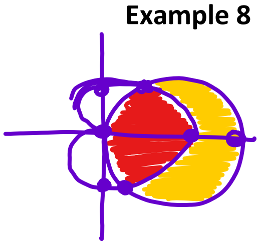
$0 = 1 + 2\cos\theta$
$-1 = 2\cos\theta$
$-\frac{1}{2} = \cos\theta$
$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

$$A = \frac{1}{2}\int_{2\pi/3}^{4\pi/3}(1+2\cos\theta)^2\,d\theta = 544$$

Example 7 Find the area between the loops of the graph of $r = 1 + 2\cos\theta$

$$\frac{1}{2}\int_{-2\pi/3}^{2\pi/3}(1+2\cos\theta)^2\,d\theta - 544 \quad \text{OR} \quad \frac{1}{2}\int_0^{2\pi}(1+2\cos\theta)^2\,d\theta - 2(544) = 8\ 338$$

Example 8 Find the area of the common interior of $r = 3\cos\theta$ and $r = 1 + \cos\theta$



$3\cos\theta = 1 + \cos\theta$
$2\cos\theta = 1$
$\cos\theta = 1/2$
$\theta = \frac{\pi}{3}, -\frac{\pi}{3}$

Area circle − Area of crescent

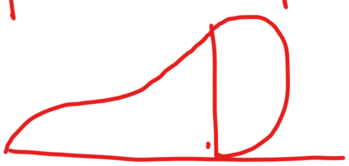
$$\frac{1}{2}\int_0^{\pi}(3\cos\theta)^2\,d\theta - \frac{1}{2}\int_{-\pi/3}^{\pi/3}\left((3\cos\theta)^2 - (1+\cos\theta)^2\right)d\theta$$

$= 3\ 927$

Example 9 A polar curve is defined by the equation $r = \theta + \sin(2\theta)$ for $0 \le \theta \le \pi$.

a) Find the area bounded by the curve and the x-axis. Calculator active

① put calculator in polar mode $r = \theta + \sin(2\theta)$ → window $0 < \theta \le \pi$ → ZOOMFIT



$$A = \frac{1}{2}\int_0^{\pi}(\theta + \sin(2\theta))^2\,d\theta = 4\ 382$$

b) Find the angle θ that corresponds to the point on the curve where $x = -2$.

$-2 = r\cos\theta$
$-2 = (\theta + \sin 2\theta)\cos\theta$

$\theta = 2\ 786$

c) For $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, $\frac{dr}{d\theta}$ is negative. What does this say about the graph on this interval?

$\frac{dr}{d\theta}$ → r is decreasing and since $r > 0$ between $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$ the graph is approaching the origin.

consider

d) At what angle θ in the interval $0 \le \theta \le \frac{\pi}{2}$ is the curve farthest away from the origin? Justify your answer.

$\frac{dr}{d\theta} = 0$ and endpts.

$\frac{dr}{d\theta} = 1 + \cos(2\theta)\,2$
$0 = 1 + 2\cos(2\theta)$
$\cos(2\theta) = -\frac{1}{2}$

$2\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \ldots$
$\frac{2\theta}{2} = \frac{\frac{2\pi}{3}}{2}, \frac{\frac{4\pi}{3}}{2}$
$\theta = \frac{\pi}{3}, \frac{2\pi}{3}$, etc.

θ	$r = \theta + \sin(2\theta)$
0	0
$\frac{\pi}{3}$	$\frac{\pi}{3} + \sin\frac{2\pi}{3} \to 1\ 57. + \frac{\sqrt{3}}{2} \to 1\ 913$
$\pi/2$	$\frac{\pi}{2} \approx 1\ 57\ldots$

at $\theta = \frac{\pi}{3}$