BC Calculus Representing Functions as Power Series Notesheet **Name:** Z

In this section, we will learn how to represent certain types of functions as the sum of a power series by modifying geometric series or by integrating and differentiating these types of series. This is a useful strategy for integrating functions that do not have easy antiderivatives, for solving differential equations, and for approximating functions using polynomials. Scientists do this to simplify the expressions they deal with; computer scientists do this to represent functions on calculators and computers. To help us get started, recall the following information about geometric series:

Geometric Series Test (GST) and the Sum of a Geometric Series

A geometric series is in the form

$$\sum_{n=0}^{\infty} a \cdot r^n \text{ or } \sum_{n=1}^{\infty} a \cdot r^{n-1},\ a \neq 0$$

The geometric series diverges if $|r| \geq 1$.

If $|r| < 1$, the series converges to the sum $S = \frac{a_1}{1-r}$.

Where a_1 is the first term, regardless of where n starts, and r is the common ratio.

Example 1 If the first term of a geometric series is 2 and $r = \frac{1}{3}$, write the first 4 terms of the geometric series, the general term of the series, and find its sum.

$2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}$

$2\left(\frac{1}{3}\right)^n,\ n \geq 0$

or $2\left(\frac{1}{3}\right)^{n-1},\ n \geq 1$

$S = \frac{2}{1-\frac{1}{3}} = \frac{2}{2/3}$

$S = 3$

Example 2 Express the function $\boxed{f(x) = \frac{1}{1-x}}$ as the sum of a power series and find the interval of convergence. Include the first 4 nonzero terms and the general term. → sum of a geometric series

$a_1 = 1$

$\underline{r = x}$

$1, x, x^2, x^3$

$\sum_{n=1}^{\infty} x^{n-1}$ or $\sum_{n=0}^{\infty} x^n$

$\underline{r = x}$

$|r| < 1$

$|x| < 1$

$-1 < x < 1$

Example 3 Express the function $f(x) = \frac{x}{1+x^2}$ as the sum of a power series and find the interval of convergence.

non-negotiable →

$a_1 = x$

$r = -x^2$

$\sum_{n=0}^{\infty} x(-x^2)^n$

↓

$x(-1)^n(x^2)^n$

$\boxed{\sum_{n=0}^{\infty} (-1)^n x^{2n+1}}$

$|-x^2| < 1$

$-1 < x < 1$

$S = \frac{x}{1-(-x^2)}$

$x, -x^3, x^5, -x^7$

Example 4 Find a power series representation for $f(x) = \frac{x^3}{1+2x}$. Include the first 4 nonzero terms and the general term.

$a_1 = x^3$

$r = -2x$

$\sum_{n=0}^{\infty} x^3(-2x)^n$

$x^3(-1)^n(2x)^n$

$\sum_{n=0}^{\infty} (-1)^n 2^n x^{n+3}$

$|-2x| < 1$

$-\frac{1}{2} < x < \frac{1}{2}$

1st 4 → $x^3, -2x^4, 4x^5, -8x^6$

Example 5 Find a power series representation for $f(x) = \frac{1}{2+x}$. Include the first 4 nonzero terms and the general term.

$\frac{1/2}{2/2 + x/2} \rightarrow \frac{1/2}{1-(-x/2)}$

$a_1 = 1/2$

$r = -\frac{x}{2}$

$|-\frac{x}{2}| < 1$

$\frac{1}{2}\left(-\frac{x}{2}\right)^n$

$-2 < x < 2$

$\frac{1}{2}(-1)^n\left(\frac{1}{2}\right)^n x^n$

$\frac{1}{2}, -\frac{x}{4}, \frac{x^2}{8}, -\frac{x^3}{16}$

$= \left(\frac{1}{2}\right)^{n+1}(-1)^n x^n$

Sometimes we cannot center a function at $x = 0$. In this case, we can try to rewrite the function with a new center showing.

Example 6 Find a power series representation for $f(x) = \frac{1}{x}$. Include the first 4 terms and the general term.

$\frac{1}{1-(1-x)}$

$\left\{\begin{array}{l} 1 - ? = x \\ 1 - x = ? \end{array}\right\}$

$|r| \rightarrow |1-x| < 1$

$0 < x < 2$

$a_1 = 1$ $r = 1 - x$

$1, 1-x, (1-x)^2, (1-x)^3$

$\sum_{n=0}^{\infty} 1(1-x)^n$

Differentiation and Integration of Power Series

The sum of a power series is a function $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ whose domain is the interval of convergence of the series. We would like to be able to differentiate and integrate such functions, and the following theorem says that we can do so by differentiating or integrating each individual term in the series, just as we would for a polynomial. This is called term-by-term differentiation and integration.

Term-by-Term Differentiation and Integration

If the power series $\sum a_n(x-c)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots = \sum_{n=0}^{\infty} a_n(x-c)^n$$

Is differentiable (and therefore continuous) on the interval $(c-R, c+R)$. The derivative is given by:

$$f'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots = \sum_{n=0}^{\infty} na_n(x-c)^{n-1}$$

And the integral is given by

$$\int f(x)dx = C + a_0(x-c) + \frac{a_1(x-c)^2}{2} + \frac{a_2(x-c)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} \frac{a_n(x-c)^{n+1}}{n+1}$$

Note: The radii of convergence of the power series for the derivative and integral are the same as $f(x)$, but the interval of convergence may not be the same because the endpoints may or may not be included.

$\frac{a_1}{1-r}$

Example 7 Find a power series representation for $f(x) = \frac{1}{(1-x)^2}$ centered at $x = 0$ using the fact that $\int f(x)dx = \frac{1}{1-x} + C$.

Find the power series for this and deriv it

$a_1 = 1$
$r = x$

$\sum_{n=0}^{\infty} 1\, x^n \rightarrow \frac{d}{dx}\sum_{n=0}^{\infty} 1\, x^n$

$\sum_{n=1}^{\infty} n\, x^{n-1}$

Instead of looking at general, expand $\frac{1}{1-x}$

$1, x, x^2, x^3, x^4$

$\frac{d}{dx} \rightarrow 0, 1, 2x, 3x^2, 4x^3$

Example 8 Find a power series representation for $f(x) = \ln(1-x)$ centered at $x = 0$ given that $f'(x) = -\frac{1}{1-x}$

$a_1 = -1$
$r = x$

Power series for $f'(x)$

$(-1)\, x^n \rightarrow$ so $\int (-1)\, x^n$

$\frac{-1}{n+1} x^{n+1}$

$S = \frac{-1}{1-x}$

$-1, -x, -x^2, -x^3, -x^4 \rightarrow \int$ these terms $\rightarrow -x, -\frac{x^2}{2}, -\frac{x^3}{3}$

Example 9 Let f be the function given by $f(x) = \frac{2x}{1+x^2}$

$S = \frac{2x}{1-(-x^2)}$

a) Write the first four nonzero terms and the general term of the series for f about $x = 0$.

$a_1 = 2x$
$r = -x^2$

$2x, -2x^3, 2x^5, -2x^7$

actual series $\rightarrow \sum_{n=0}^{\infty} 2x\,(-x^2)^n \rightarrow 2x\,(-1)^n\,(x^2)^n \rightarrow 2x\,(-1)^n\, x^{2n} \rightarrow (-1)^n\, 2\, x^{2n+1}$

b) Does the series found in part a), when evaluated at $x = 1$, converge to $f(1)$? Explain why or why not.

$|-x^2| < 1$

$-1 < x < 1$

No, because $x = 1$ is not included in the interval of convergence

c) The derivative of $\ln(1+x^2)$ is $\frac{2x}{1+x^2}$. Write the first 4 terms of the series for $\ln(1+x^2)$ about $x = 0$.

$\frac{d}{dx}$ $f(x)$

come up w/ series (which we did)

1st 4 terms from (a)

$\int 2x, -2x^3, +2x^5, -2x^7$

$x^2, -\frac{1}{2}x^4, \frac{1}{3}x^6, -\frac{1}{4}x^8$

$\int (-1)^n\, 2\, x^{2n+1}$

$\frac{(-1)^n\, 2\, x^{2n+2}}{2n+2}$

d) Use the series found in part c) to find a rational number A where $\left|A - \ln\left(\frac{5}{4}\right)\right| < \frac{1}{100}$. Justify your answer.

$A = \frac{7}{32}$

error bound

A = sum $\rightarrow$ implies $\ln\left(\frac{5}{4}\right)$ = partial sum at some value of n

$\ln(1+x^2) = \ln\left(\frac{5}{4}\right) \rightarrow \ln\left(1+\frac{1}{4}\right) \rightarrow x^2 = \frac{1}{4} \rightarrow x = \frac{1}{2}$

$\left(\frac{1}{2}\right)^2, -\frac{1}{2}\left(\frac{1}{2}\right)^4, \frac{1}{3}\left(\frac{1}{2}\right)^6$

$\boxed{\frac{1}{4} + -\frac{1}{32}}$

a_3 $\frac{1}{192}$