BC Calculus Sequence Convergence/Divergence Notesheet **Name: ___________________________________**

A sequence is a list of things generated by a rule

pattern

explicit Rule

recursive

More formally, a sequence is a function whose domain is the set of positive integers, or natural numbers, n such that $n \in \mathbb{N} = \{1, 2, 3, \dots\}$. The range of the function are called the terms in the sequence,

$4, 8, 12, \quad , 4n$ $\qquad a_1, a_2, a_3, \dots a_n$ $\qquad \{4n\}$ $\qquad a_n = a_{n-1} + 4$

Where a_n is called the nth term (or rule of the sequence), and we denote the sequence by $\{a_n\}$.

The sequence can be expressed by either

1. An ample number of terms in the sequence, separated by commas
2. An recursive function including a first term and a rule using the previous term of the sequence.
3. A rule for the sequence given as an explicit function set off in curly braces.

Example 1

The sequence $2, 4, 6, 8, \dots$ is the sequence of even numbers. Express the same sequence as a rule of a non-negative integer n. The sequence $1, 3, 5, 7, \dots$ is the sequence of odd numbers. Express the same sequence as a rule of a non-negative integer n. How many in the list are needed to establish the "rule" in the absence of the explicitly-stated rule?

$2, 4, 6, 8, \dots \qquad a_n = \{2n\}$

$1, 3, 5, 7, \dots \qquad a_n = \{2n - 1\}$

Note: When given a sequence as a list, the first term is usually designated to be associated with $n = 1$. This is because we are using n as a counting number.

We will be primarily interested in what happens to the sequence for increasingly large values of n.

Example 2

If $a_n = \left\{\frac{4n}{3+2n}\right\}$, list out the first five terms, then estimate $\lim_{n\to\infty} a_n = \boxed{2}$

$a_1 = \frac{4}{5} \qquad a_2 = \frac{8}{7} \qquad a_3 = \frac{12}{9} = \frac{4}{3} \qquad a_4 = \frac{16}{11} \qquad a_5 = \frac{20}{13}$

Possibilities for Sequences as $n \to \infty$

Let $\{a_n\}$ be a sequence of real numbers.

- If $\lim_{n\to\infty} a_n = \infty$, then $\{a_n\}$ diverges to infinity.
- If $\lim_{n\to\infty} a_n = -\infty$, then $\{a_n\}$ diverges to negative infinity
- If $\lim_{n\to\infty} a_n = c$, a finite real number, then $\{a_n\}$ converges to c.
- If $\lim_{n\to\infty} a_n$ oscillates between two fixed numbers, then $\{a_n\}$ diverges by oscillation

Factorial

$n!$ Is read as "n factorial." It is defined recursively as $n! = n(n-1)!$ or as

$$n! = n(n-1)! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

For example, $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

Example 3 Determine whether the following sequences converge or diverge

a_n

a) $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots$

$\lim_{n\to\infty} a_n = 1 \to$ a_n converges to 1

b) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$

$\lim_{n\to\infty} a^n = 0$ a_n converges to 0

c) $a_n = \{3 + (-1)^n\}$

oscillates diverges

d) $a_n = \left\{\frac{n}{1-2n}\right\}$

$\lim_{n\to\infty} a_n = -\frac{1}{2}$ converges

e) $a_n = \left\{\frac{\ln n}{n}\right\}$

converges to 0

f) $a_n = \left\{\frac{n!}{(n+2)!}\right\}$ simplify ◡

$= \frac{\cancel{n!}}{(n+2)(n+1)\cancel{n!}} \to \frac{1}{(n+2)(n+1)} \to 0$ converge

g) $a_n = \left\{\frac{2n!}{(n-1)!}\right\}$

$= \frac{2n\cancel{(n-1)!}}{\cancel{(n-1)!}} \to 2n \to$ diverges to ∞

h) $a_n = \left\{\frac{n + (-1)^n}{n}\right\}$

$a_n = \left\{\frac{n}{n} + \frac{(-1)^n}{n}\right\}$ converges

$\lim_{n\to\infty} a_n = 1 + 0 = 1$

i) $a_n = \left\{\frac{(-1)^n(n-1)}{n}\right\}$

oscillating divergent

j) $a_n = \left\{\frac{2^n}{(n+1)!}\right\}$

Factorials are very BIG ◡

$\lim_{n\to\infty} a_n = 0$ convergent

k) $a_n = \left\{\left(1 + \frac{1}{n}\right)^n\right\}$

? $\lim_{n\to\infty}$ $n \ln\left(1 + \frac{1}{n}\right)$

$\to \frac{\ln\left(1+\frac{1}{n}\right)}{1/n} \to \frac{0}{0} \to$ L'H $\to \frac{\frac{1}{1+\frac{1}{n}}\left(-\frac{1}{n^2}\right)}{\left(-\frac{1}{n^2}\right)} \to \frac{1}{1+\frac{1}{n}} \to 1$

undo the ln so $e^1 = e$

l) $a_n = \left\{\frac{(2n)!}{(2n-2)!}\right\}$

$\frac{2n(2n-1)\cancel{(2n-2)!}}{\cancel{(2n-2)!}} \to \infty$ divergent

Sometimes, we have to write the rule of a sequence as a function of n from a pattern.

Example 4 Determine whether the following sequences converge or diverge

a) $3, 8, 13, 18, \ldots$

$a_n = \{5n - 2\}$ diverge

b) $5, -15, 45, -135, \ldots$

$a_n = \{(-3)^{n-1} 5\}$ diverging

c) $1, 4, 9, 16, 25, \ldots$

$a_n = \{n^2\}$ diverging

d) $4, 10, 28, 82, \ldots$

$a_n = \{3^n + 1\}$ diverge

e) $\frac{2}{1}, \frac{3}{3}, \frac{4}{5}, \frac{5}{7}, \frac{6}{9}, \ldots$

$a_n = \left\{\frac{n+1}{2n-1}\right\}$ convergent to $\frac{1}{2}$

f) $\ln 1, \ln 2, \ln 4, \ln 8, \ldots$

$a_n = \{\ln(2^{n-1})\}$ divergent