BC Calculus Series Convergence/Divergence A Notesheet **Name:** ___________________________________

A series is the sum of the terms in a sequence. Finite sequences and series have defined first and last terms, whereas infinite sequences and series continue indefinitely. Informally, a series is the result of adding any number of terms from a sequence together: $a_1 + a_2 + a_3 + \cdots$. A series can be written more succinctly by using the summation symbol.

For infinite series, we can look at the sequence of partial sums, that is, looking to see what the sums are doing as we add additional terms. In general, the nth partial sum of a series is denoted S_n. This can be explored on a calculator by adding sequential terms to the aggregate sum. $S_1 = a_1$ / $S_2 = a_1 + a_2$ / $S_3 = a_1 + a_2 + a_3$ ..

Example 1 For both $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n^2}$, generates the sequence of partial sums $S_1, S_2, S_3, \dots, S_n$, for each, then determine if the series converge or diverge. Where else have we seen something like this before?

$S_n = \sum_{n=1}^{\infty} \frac{1}{n}$ ∞

$= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} +$

For visual calc says

$S_{10} = 2.93$

$S_{100} = 5.19$

$S_{1000} = 7.49$

diverges

$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \dots$

$S_{10} = 1.55$

$S_{100} = 1.635$

$S_{1000} = 1.644$

→ seems to be more of a converging series

Example 2 Given the series

$$\sum_{n=1}^{\infty} \frac{3}{2^n} = \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \frac{3}{16} + \frac{3}{32} + \frac{3}{64} + \frac{3}{128} + \frac{3}{256} + \cdots,$$

Find the first 10 terms of the sequence of partial sums, and list them below, $S_1, S_2, S_3, \dots, S_{10}$. Based on this sequence of partial sums, do you think the series converges or diverges? To what? (Hint: first rewrite the rule of the sequence so that it looks like an exponential function of n)

$S_1 = \frac{3}{2}$ $S_5 = 2.90625$ $S_{10} = 2.997$ $S_{100} = 3$ (probably 2.99999999)

converges to 3 $3\left(\frac{1}{2}\right)^n$ → geometric $S_\infty = \frac{a_1}{1-r}$

Example 3 Given the series

$$\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n = \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \frac{81}{16} + \frac{243}{32} + \cdots,$$

Find the first 5 terms of the sequence of partial sums, and list them below, $S_1, S_2, S_3, \dots, S_5$. Based on this sequence of partial sums, do you think the series converges or diverges? To what?

diverge since $r = \frac{3}{2}$ is larger than 1

We are now going to look at several families of infinite series and several tests that will help us determine whether they converge or diverge. For some that converge, we might be able to give the actual sum, or an interval in which we know the sum will be. For others, simply knowing that they converge will have to suffice.

Geometric Series Test (GST)

A geometric series is in the form

$$\sum_{n=0}^{\infty} a \cdot r^n \text{ or } \sum_{n=1}^{\infty} a \cdot r^{n-1}, \quad a \neq 0$$

The geometric series diverges if $|r| \geq 1$.

If $|r| < 1$, the series converges to the sum $S = \frac{a_1}{1-r}$.

Where a_1 is the first term, regardless of where n starts, and r is the common ratio.

Example 4 Using the GST, determine whether each series converges or diverges. If it converges, find the sum.

a) $\sum_{n=1}^{\infty} \frac{3}{2^n} \to 3\left(\frac{1}{2}\right)^n$

$r = \frac{1}{2}$

$\left|\frac{1}{2}\right| < 1$, so by GST $\sum_{n=1}^{\infty} \frac{3}{2^n}$ converge

converges to $\frac{3/2}{1-1/2} = \frac{3/2}{1/2} = 3$

b) $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$ $r = \frac{3}{2}$

$\left|\frac{3}{2}\right| \geq 1$

$\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$ diverge by GST

c) $\sum_{n=2}^{\infty} 3\left(-\frac{1}{2}\right)^n$

$r = \frac{-1}{2}$

$\left|-\frac{1}{2}\right| < 1$

so $\sum_{n=2}^{\infty} 3\left(-\frac{1}{2}\right)^n$ converges

to $\frac{a_2}{1-r} = \frac{3/4}{3/2} = \frac{1}{2}$

nth Term Test for Divergence

If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Note: This does not say that if $\lim_{n\to\infty} a_n = 0$, then the series converges. This test can only be used to prove that a series diverges, hence the name. If $\lim_{n\to\infty} a_n \neq 0$, then this test does not tell us anything, is inconclusive, does not work, fails, etc. We must use another test. This test can be a great time-saver. Always perform it first.

Example 5 Use the nth term test to determine whether the following series diverge.

a) $\sum_{n=1}^{\infty} \frac{2n+3}{3n-5}$ → this series will diverge by nth term test

$a_n = \frac{2n+3}{3n-5}$

$\lim_{n\to\infty} a_n = \frac{2}{3} \neq 0$

b) $\sum_{n=1}^{\infty} \frac{n!}{2n!+1}$

$\lim_{n\to\infty} \frac{n!}{2n!+1} = \frac{1}{2} \to$ by nth term test diverges

since $\frac{1}{2} \neq 0$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n - 2}{3^n}$

$\lim_{n\to\infty} \frac{3^n-2}{3^n} = 1 \neq 0$ so by n^{th} term test this series diverges

d) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(1.1)^n}$

n^{th} term test is inconclusive since $\lim_{n\to\infty} \frac{1}{(1.1)^n} = 0$

so I have to use a different test

$\left(\frac{1}{1.1}\right)^n$ $\quad |r = \frac{1}{1.1}| < 1$ ✓

so by GST, it converges

Telescoping Series

A series such as $\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots$ is called a telescoping series because it collapses to one term or just a few terms. If a series collapses to a finite sum, then it converges by the Telescoping Series Test. Write out terms of the series until both the start and ending terms cancel out. Then add the terms that do not cancel out to find the sum of the series.

Example 6 Determine whether the following series converge or diverge. IF they converge, find their sum.

a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{2n+1}-\frac{1}{2n+3}\right)$

$$\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{9}\right)+\cdot \quad = \frac{1}{3}$$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$$\frac{A}{n}+\frac{B}{n+1}=\frac{1}{n(n+1)}$$

$$A(n+1)+Bn = 1$$

if $n=0 \rightarrow A=1$

if $n=-1 \rightarrow B=-1$

$$\sum_{n=1}^{\infty} \frac{1}{n}-\frac{1}{n+1}$$

$$\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots = 1$$

c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4n+3}$ partial fractions

$$\sum_{n=1}^{\infty} \frac{-\frac{1}{2}}{n+3}+\frac{\frac{1}{2}}{n+1}$$

$$\sum_{n=1}^{\infty} -\frac{1}{2}\cdot\frac{1}{n+3}+\frac{1}{2}\cdot\frac{1}{n+1}$$

$$\left(-\cancel{\tfrac{1}{8}}+\frac{1}{4}\right)+\left(-\cancel{\tfrac{1}{10}}+\frac{1}{6}\right)$$

$$+\left(-\cancel{\tfrac{1}{12}}+\cancel{\tfrac{1}{8}}\right)+\left(-\cancel{\tfrac{1}{14}}+\cancel{\tfrac{1}{10}}\right)$$

$$+\left(-\frac{1}{16}+\cancel{\tfrac{1}{12}}\right)+\left(-\frac{1}{18}+\cancel{\tfrac{1}{14}}\right)$$

$$\frac{1}{4}+\frac{1}{6}$$

$$\frac{3+2}{12}=\frac{5}{12}$$

Integral Test

If f is decreasing, continuous, and positive for $x \geq 1$ and $a_n = f(x)$, then

$$\sum_{n=1}^{\infty} a_n \text{ and } \int_1^{\infty} f(x)dx$$

Either both converge or diverge.
Note 1: This does not mean that the series converges to the value of the definite integral.
Note 2: The function need only be decreasing for all $x > k$ for some $k \geq 1$.

Example 7 Determine whether the following series converge or diverge.

a) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ $\quad \int_1^{\infty} \frac{x}{x^2+1}dx$

$u = x^2+1$

$\frac{du}{dx} = 2x$

$\frac{1}{2}du = x\,dx$

$\lim_{b\to\infty} \int_1^b \frac{x}{x^2+1}dx$

$\frac{1}{2}\int \frac{1}{u}du$

$\frac{1}{2}\ln|u| \to \lim_{b\to\infty} \frac{1}{2}\ln|x^2+1|\Big|_1^b \to \lim_{b\to\infty} \frac{1}{2}\ln|b^2+1| - \frac{1}{2}\ln|1^2+1| \Rightarrow \infty$

diverges so the series will also diverge by Integral Test

b) $\sum_{n=1}^{\infty} \frac{1}{n^2+1} \to \int_1^{\infty} \frac{1}{x^2+1}dx$

$\lim_{b\to\infty} \int_1^b \frac{1}{x^2+1}dx = \lim_{b\to\infty} \tan^{-1}x\Big|_1^b$

$\lim_{b\to\infty} \tan^{-1}b - \tan^{-1}(1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

therefore the series converges by Integral Test (probably do NOT $\pi/4$)

P-Series Test

A series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p}$$

Is called a p-series, where p is a positive constant. If $p = 1$, the series is called the harmonic series.

- If $p \leq 1$ the series will diverge.
- If $p > 1$ the series will converge.

Note: If the p-series converges and starts at $n = 1$, we cannot find its sum using $\frac{1}{p-1}$ like we could with p-series integrals.

Example 8 Use the ~~nth term~~ p-series test to determine whether the following series diverge.

a) $\boxed{\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}} \to \frac{1}{n^{1.5}}$

$p = 1.5$ $\quad 1.5 > 1$

will converge by p-series

b) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n}$ $\quad \frac{1}{n^{.5}}$

$p = .5$ $\quad .5 \leq 1$

will diverge by p-series