BC Calculus Series Convergence/Divergence B Notesheet **Name:** ______________________

Direct Comparison Test (DCT)

If $a_n \geq 0$ and $b_n \geq 0$,

If $\sum_{n=1}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$, then $\sum_{n=1}^{\infty} a_n$ converges.

If $\sum_{n=1}^{\infty} a_n$ diverges and $0 \leq a_n \leq b_n$, then $\sum_{n=1}^{\infty} b_n$ diverges.

Note: You must state/show the inequality when stating the conclusion of this test.

Example 1 Determine whether the following series converge or diverge.

a) $\sum_{n=1}^{\infty} \frac{n^3}{n^3+1}$

$\lim_{n\to\infty} \frac{n^3}{n^3+1} = 1$ n^{th} term diverge

b) $\sum_{n=1}^{\infty} \frac{1}{n^3}$

p-series
$p = 3$
$3 > 1$
converge

c) $\sum_{n=1}^{\infty} \frac{1}{3^n+2}$ by DCT, will also converge

$0 < \frac{1}{3^n+2} \leq \frac{1}{3^n}$

$\downarrow$

$\left(\frac{1}{3}\right)^n$

GST w/ $r = \frac{1}{3}$
converge

d) $\sum_{n=4}^{\infty} \frac{1}{\sqrt{n}-1}$

For $n \geq 4$

$\frac{1}{\sqrt{n}-1} \geq \frac{1}{\sqrt{n}} \geq 0$

Divergent as well by DCT

$\downarrow$

$\frac{1}{n^{\frac{1}{2}}} \to$ p-series $p = \frac{1}{2} \leq 1$
divergent

e) $\sum_{n=1}^{\infty} \frac{|\cos n|}{2^n} \to$ by DCT, converge

$n \geq 1$

$0 \leq \frac{|\cos n|}{2^n} \leq \frac{1}{2^n}$

$\downarrow$

$\left(\frac{1}{2}\right)^n$

$\downarrow$

Geo Series Test
$r = \frac{1}{2}$ $0 < \frac{1}{2} < 1$
converges

f) $\sum_{n=2}^{\infty} \frac{1}{n^4-10}$

$\frac{1}{n^4-10} \geq \frac{1}{n^4} \geq 0$

$n \geq 2$

p-series $p = 4$ converge
Inconclusive

$\lim_{n\to\infty} \frac{\frac{1}{n^4-10}}{\frac{1}{n^4}}$

$\lim_{n\to\infty} \frac{n^4}{n^4-10} = 1$

by LCT (lim.'s), $\sum_{n=2}^{\infty} \frac{1}{n^4-10}$ converges

Limit Comparison Test (LCT)

If $a_n \geq 0$ and $b_n \geq 0$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ or $\lim_{n\to\infty} \frac{b_n}{a_n} = L$, where L is both finite and positive, then the two series

$$\sum_{n=1}^{\infty} a_n \ or \ \sum_{n=1}^{\infty} b_n$$

either both converge or both diverge.

Note: You must show the limit when stating the conclusion of this test.

Example 2 Determine whether the following series converge or diverge.

a) $\sum_{n=1}^{\infty} \frac{1}{3n^2 - 4n + 5}$ → Limit comparison

converge

$$\lim_{n\to\infty} \frac{\frac{1}{3n^2-4n+5}}{\frac{1}{n^2}} = \frac{1}{3}$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ p-series

p = 2

2 > 1

converge

b) $\sum_{n=1}^{\infty} \frac{n^4}{4n^5 - n^3 + 7}$ LCT

diverge

$$\lim_{n\to\infty} \frac{\frac{n^4}{4n^5-n^3+7}}{\frac{1}{n}}$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ → Harmonic series

diverge

c) $\sum_{n=2}^{\infty} \frac{1}{n^3 - 2}$ → Limit comparison test → converge

$$\lim_{n\to\infty} \frac{\frac{1}{n^3-2}}{\frac{1}{n^3}} = 1$$

$\sum_{n=2}^{\infty} \frac{1}{n^3}$ → converge

p-series

p = 3

3 > 1

d) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n - 2}}$ by LCT, diverge

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{3n-2}}}{\frac{1}{\sqrt{n}}} = \frac{1}{\sqrt{3}}$$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ → p-series

$p = \frac{1}{2}$ $\frac{1}{2} \leq 1$

divergent

Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series of nonzero terms.

$\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$

The ratio test is inconclusive if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$

Ratio is very helpful for exponentials and factorials

Example 3 Determine whether the following series converge or diverge.

a) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

$\lim_{n\to\infty} \left| \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} \right|$

$\lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n} \right|$

$\lim_{n\to\infty} \left| \frac{2^n 2}{(n+1)n!} \frac{n!}{2^n} \right|$

$\lim_{n\to\infty} \left| \frac{2}{n+1} \right| = 0$

$0<1$

converges by Ratio test

b) $\sum_{n=1}^{\infty} \frac{n^2(3^n+1)}{2^n}$ Ratio Test divergent

$\lim_{n\to\infty} \left| \frac{\frac{(n+1)^2(3^{n+1}+1)}{2^{n+1}}}{\frac{n^2(3^n+1)}{2^n}} \right| \to \lim_{n\to\infty} \left| \frac{(n+1)^2(3^{n+1}+1)\,2^n}{2^{n+1}\,n^2(3^n+1)} \right|$

$= \frac{3}{2} > 1$

c) $\sum_{n=1}^{\infty} \frac{(n+1)!}{3^n}$ divergent ratio test

$\lim_{n\to\infty} \left| \frac{(n+2)!}{3^{n+1}} \cdot \frac{3^n}{(n+1)!} \right|$

$\lim_{n\to\infty} \left| \frac{(n+2)(n+1)!\,3^n}{3^n\,3\,(n+1)!} \right| \to \lim_{n\to\infty} \left| \frac{n+2}{3} \right| = \infty$

d) $\sum_{n=1}^{\infty} \frac{3^{n-1}}{n\,2^n}$ ratio test divergent

$\lim_{n\to\infty} \left| \frac{3^n}{(n+1)2^{n+1}} \cdot \frac{n\,2^n}{3^{n-1}} \right|$

$\lim_{n\to\infty} \left| \frac{\cancel{3^n}\,\cancel{n}\,\cancel{2^n}\,1\,3}{\cancel{(n+1)}\,\cancel{2^n}\,2\,\cancel{3^n}\,\cancel{3^{-1}}} \right| = \frac{3}{2} > 1$

Root Test

Let $\sum_{n=1}^{\infty} a_n$ be a series of nonzero terms.

$\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$

The root test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$

Example 4 Determine whether the following series converge or diverge.

a) $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$ $\lim_{n\to\infty} \sqrt[n]{\left| \frac{e^{2n}}{n^n} \right|}$

$= \lim_{n\to\infty} \frac{e^2}{n} = 0 \quad 0<1$

Root test converge

b) $\sum_{n=1}^{\infty} \left(\frac{3n+4}{2n}\right)^n$ $\lim_{n\to\infty} \sqrt[n]{\left| \left(\frac{3n+4}{2n}\right)^n \right|}$

$\lim_{n\to\infty} \frac{3n+4}{2n} = \frac{3}{2} > 1$

Root test diverge