BC Calculus Series Convergence/Divergence B Notesheet **Name:** ___________________________________

Direct Comparison Test (DCT)

If $a_n \geq 0$ and $b_n \geq 0$,

If $\sum_{n=1}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$, then $\sum_{n=1}^{\infty} a_n$ converges.

If $\sum_{n=1}^{\infty} a_n$ diverges and $0 \leq a_n \leq b_n$, then $\sum_{n=1}^{\infty} b_n$ diverges.

Note: You must state/show the inequality when stating the conclusion of this test.

Example 1 Determine whether the following series converge or diverge.

a) $\sum_{n=1}^{\infty} \frac{n^3}{n^3+1}$

by n^{th} term test, this diverges

$\lim_{n\to\infty} \frac{n^3}{n^3+1} = 1 \neq 0$

b) $\sum_{n=1}^{\infty} \frac{1}{n^3}$

p-series $p = 3$ $3 > 1$

converges

c) $\sum_{n=1}^{\infty} \frac{1}{3^n+2}$

$0 < \frac{1}{3^n+2} < \frac{1}{3^n} \rightarrow \left(\frac{1}{3}\right)^n$

Must converge by Direct comparison test

G series w/ $r = \frac{1}{3}$

$\left|\frac{1}{3}\right| < 1$

converge

d) $\sum_{n=4}^{\infty} \frac{1}{\sqrt{n}-1}$

$\frac{1}{\sqrt{n}-1} \geq \frac{1}{\sqrt{n}} \geq 0$

by Direct comparison Test this diverges

p-series $p = \frac{1}{2}$

$\frac{1}{2} \leq 1$

diverges

e) $\sum_{n=1}^{\infty} \frac{|\cos n|}{2^n}$

$0 \leq \frac{|\cos n|}{2^n} \leq \frac{1}{2^n} \rightarrow \left(\frac{1}{2}\right)^n$

by DCT, this converges

GST w/ $r = \frac{1}{2}$

$\left|\frac{1}{2}\right| < 1$

converges

f) $\sum_{n=2}^{\infty} \frac{1}{n^4-10}$

$\frac{1}{n^4-10} \geq \frac{1}{n^4} \geq 0$

$\boxed{\text{Just b/c the smaller converges, does not imply the larger converges}}$

converges

INconclusive

use Limit comparison Test

$\lim_{n\to\infty} \frac{\frac{1}{n^4-10}}{\frac{1}{n^4}} = 1$

so converges by LCT.

Limit Comparison Test (LCT)

If $a_n \geq 0$ and $b_n \geq 0$, and $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = L$ or $\lim\limits_{n\to\infty} \frac{b_n}{a_n} = L$, where L is both finite and positive, then the two series

$$\sum_{n=1}^{\infty} a_n \ or \ \sum_{n=1}^{\infty} b_n$$

either both converge or both diverge.

Note: You must show the limit when stating the conclusion of this test.

Example 2 Determine whether the following series converge or diverge.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^2 - 4n + 5}$ LCT converge

$$\lim_{n\to\infty} \frac{\frac{1}{3n^2-4n+5}}{\frac{1}{n^2}} = \frac{1}{3}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ → p-series

$p = 2$

$2 > 1$

converges

b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{4n^5 - n^3 + 7}$ LCT, diverge

$$\lim_{n\to\infty} \frac{\frac{n^4}{4n^5-n^3+7}}{\frac{1}{n}} = \frac{1}{4}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (Harmonic-series p-series, w/ $p=1$)

c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3 - 2}$ LCT, converges

$$\lim_{n\to\infty} \frac{\frac{1}{n^3-2}}{\frac{1}{n^3}} = 1$$

$\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3}$ converges

d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n - 2}}$ LCT, diverges

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{3n-2}}}{\frac{1}{\sqrt{n}}} \to \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{3n-2}} \to \sqrt{\frac{1}{3}}$$

$\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}}$ diverges

p-series

$p = \frac{1}{2}$ $\quad \frac{1}{2} \leq 1$ → diverges

Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series of nonzero terms.

$\sum_{n=1}^{\infty} a_n$ converges if $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$

The ratio test is inconclusive if $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$

Example 3 Determine whether the following series converge or diverge.

a) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

converges by ratio test

$$\lim_{n\to\infty} \left| \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} \right|$$

$$\lim_{n\to\infty} \left| \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} \right|$$

$$\lim_{n\to\infty} \left| \frac{\cancel{2^n}\,2}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{2^n}} \right|$$

$$\lim_{n\to\infty} \left| \frac{2}{n+1} \right| = 0$$, and $0<1$

b) $\sum_{n=1}^{\infty} \frac{n^2(3^n+1)}{2^n}$ a_n

remember $\frac{3^n}{2^n} \to \left(\frac{3}{2}\right)^n$

$\lim_{n\to\infty} a_n \to \infty$, the nth term test concludes divergence

If ratio were used

$$\lim_{n\to\infty} \left| \frac{(n+1)^2(3^{n+1}+1)}{\cancel{2^{n}}\,2} \cdot \frac{\cancel{2^n}}{n^2(3^n+1)} \right| = \frac{3}{2} > 1$$

c) $\sum_{n=1}^{\infty} \frac{(n+1)!}{3^n}$

$$\lim_{n\to\infty} \left| \frac{\frac{(n+2)!}{3^{n+1}}}{\frac{(n+1)!}{3^n}} \right|$$

$$\lim_{n\to\infty} \left| \frac{(n+2)!}{3^{n+1}} \cdot \frac{3^n}{(n+1)!} \right|$$

$$\lim_{n\to\infty} \left| \frac{(n+2)\cancel{(n+1)!}}{\cancel{3^n}\,3^1} \cdot \frac{\cancel{3^n}}{\cancel{(n+1)!}} \right|$$

ratio test $\to \infty$ diverges

d) $\sum_{n=1}^{\infty} \frac{3^{n-1}}{n \cdot 2^n}$

$$\lim_{n\to\infty} \left| \frac{\frac{3^n}{(n+1)2^{n+1}}}{\frac{3^{n-1}}{n\cdot 2^n}} \right|$$

$$\lim_{n\to\infty} \left| \frac{\cancel{3^n}}{(n+1)\cancel{2^n}\,2} \cdot \frac{n\,\cancel{2^n}}{\cancel{3^n}\cdot 3^{-1}} \right|$$

$$\lim_{n\to\infty} \left| \frac{3n}{2n+2} \right| = \frac{3}{2}$$

$\frac{3}{2} > 1$ ratio test says this diverges

Root Test

Let $\sum_{n=1}^{\infty} a_n$ be a series of nonzero terms.

$\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$

The root test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$

Example 4 Determine whether the following series converge or diverge.

a) $\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$

$$\lim_{n\to\infty} \sqrt[n]{\left| \frac{e^{2n}}{n^n} \right|}$$

$$\lim_{n\to\infty} \frac{e^2}{n} = 0$$

by root test, this converges

b) $\sum_{n=1}^{\infty} \left(\frac{3n+4}{2n}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{\left| \left(\frac{3n+4}{2n}\right)^n \right|}$$

$$\lim_{n\to\infty} \frac{3n+4}{2n} = \frac{3}{2}$$ diverges by root test