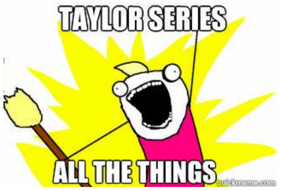
BC Calculus Taylor Series Notesheet **Name:** ____________________________________

Taylor Polynomials can be used to approximate other elementary functions such as $y = \sin x$, $y = e^x$, and $y = \ln x$.



Example 1 Find the equation of the tangent line for $f(x) = \sin x$ at $x = 0$, then use it to approximate $\sin(0.2)$. Is this an over or under approximation of $\sin(0.2)$?

$f(0) = 0$

$f'(x) = \cos x$

$f'(0) = 1$

$y - 0 = 1(x - 0)$

$y = x$ $\sin(2) \approx 2$

$f''(x) = -\sin x \rightarrow$ ccdown

$+ \mid -$; $-\pi$ 0 π

overestimate

The equation of the tangent line used in example 1 is called a **first degree Taylor polynomial**. Taylor polynomials of higher degree can be used to obtain increasingly better approximations of non-polynomial functions with a certain **radius** from a **center of approximation** $x = c$.

Definition of an nth degree Taylor Polynomial

If f has n derivatives at $x = c$, then the polynomial

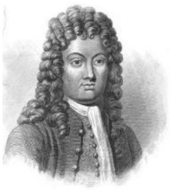
$$P_n(x) = \underbrace{f(c)}_{0!} + \underbrace{f'(c)(x-c)}_{1!} + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

$\rightarrow$ general term

$0! = 1$

$1! = 1$

is called the nth degree Taylor polynomial for f centered at c.



Note 1: A first-degree Taylor polynomial is a tangent line to f at c.

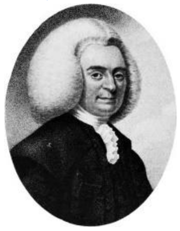
Note 2: $\frac{f^{(n)}(c)}{n!}$ is the coefficient of the $(x - c)^n$ term.

Definition of nth degree Maclaurin Polynomial $\rightarrow$ Just easy Taylor polynomials

If the Taylor polynomial is centered at 0 ($c = 0$), then

$$P_n(x) = f(c) + f'(c)(x) + \frac{f''(c)}{2!}(x)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x)^n$$

is called the nth degree Maclaurin polynomial for f.



Example 3 Find the Maclaurin polynomial of degree $n = 7$ for $f(x) = \sin x$. Then use $P_7(x)$ to approximate the value of $\sin(0.1)$ using correct notation. Find the error for your approximation.

$f(x) = \sin x = f''''(x)$ $f(0) = 0$

$f'(x) = \cos x = f^5(x)$ $f'(0) = 1$

$f''(x) = -\sin x = f^6(x)$ $f''(0) = 0$

$f'''(x) = -\cos x = f^7(x)$ $f'''(0) = -1$

$$P_7(x) = 0 + 1x + \frac{0x^2}{2!} + \frac{(-1)x^3}{3!} + \frac{0x^4}{4!} + \frac{1x^5}{5!} + \frac{0x^6}{6!} + \frac{(-1)x^7}{7!}$$

$$\boxed{\sin x \approx P_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}}$$

$$\sin(1) \approx P_7(1) = 1 - \frac{(1)^3}{3!} + \frac{(1)^5}{5!} - \frac{(1)^7}{7!}$$

$= 0998334166$ / Error is less than calculator capability

Example 4 List the first four non-zero terms and the nth term for the Maclaurin series for $f(x) = \sin x$.

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$$ for $n \geq 1$

$\cos x \approx P_7(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$ / nth term → $\frac{(-1)^{n-1}x^{2n-2}}{(2n-2)!}, n \geq 1$

Example 5 Find the Maclaurin polynomial of degree $n = 7$ for $f(x) = \cos x$. Use it to find the first four non-zero terms and the nth term for the Maclaurin series for $f(x) = \cos x$.

$f(x) = \cos x = f^4(x)$ $\quad f(0) = 1 = f^4(0)$

$f'(x) = -\sin x = f^5(x)$ $\quad f'(0) = 0 = f^5(0)$

$f''(x) = -\cos x = f^6(x)$ $\quad f''(0) = -1 = f^6(0)$

$f'''(x) = \sin x = f^7(x)$ $\quad f'''(0) = 0 = f^7(0)$

$P_7(x) = 1 + 0x - \frac{1x^2}{2!} + \frac{0x^3}{3!} + \frac{1x^4}{4!} + \frac{0x^5}{5!} - \frac{1x^6}{6!} + \frac{0x^7}{7!}$

Example 6 Find the Maclaurin polynomial of degree $n = 4$ for $f(x) = e^x$. Use it to find the first four non-zero terms and the nth term for the Maclaurin series for $f(x) = e^x$.

$f(x) = e^x = f'(x) = f''(x) = f'''(x) = f^4(x)$

$f(0) = e^0 = 1 = f'(0) = f''(0) = f'''(0) = f^4(0)$

$P_4(x) = 1 + 1x + \frac{1x^2}{2!} + \frac{1x^3}{3!} + \frac{1x^4}{4!}$ general term → $\frac{x^n}{n!}, n \geq 0$

Example 7 Find the Taylor polynomial of degree $n = 4$ for $f(x) = \ln x$ centered at $c = 1$. Then use $P_4(x)$ to approximate the value of $\ln(1.1)$.

$f(x) = \ln x \rightarrow f(1) = 0$ $\quad f^4(x) = \frac{-6}{x^4} \rightarrow f^4(1) = -6$

$f'(x) = \frac{1}{x} \rightarrow f'(1) = 1$

$f''(x) = -\frac{1}{x^2} \rightarrow f''(1) = -1$

$f'''(x) = \frac{2}{x^3} \rightarrow f'''(1) = 2$

$P_4(x) = 0 + 1(x-1) - \frac{1(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} - \frac{6(x-1)^4}{4!}$

$P_4(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$ / $\ln(1.1) \approx .1 - \frac{(.1)^2}{2} + \frac{(.1)^3}{3} - \frac{(.1)^4}{4} = .0953$

Example 8 Suppose that g is a function which has continuous derivatives, and that $g(2) = 3$, $g'(2) = -4$, $g''(2) = 7$, and $g'''(2) = -5$. Write a Taylor polynomial of degree 3 for g centered at 2.

$g(x) \approx P_3(x) = 3 - 4(x-2) + \frac{7(x-2)^2}{2!} - \frac{5(x-2)^3}{3!}$

Example 9 Let $P(x) = 7 - 3(x-4) + 5(x-4)^2 - 2(x-4)^3 + 6(x-4)^4$ be the 4th degree Taylor polynomial for the function f about $x = 4$. Assume f has derivatives of all orders for all real numbers.

a) Find $f(4)$ and $f'''(4)$ → comes from this term $-2(x-4)^3 = \frac{f'''(4)(x-4)^3}{3!} \rightarrow -2 = \frac{f'''(4)}{6} \rightarrow f'''(4) = -12$

$f(4) = 7$

b) Write the second-degree Taylor polynomial for f' about $x = 4$ and use it to approximate $f'(4.3)$.

$P(x) = -3 + 10(x-4) - 6(x-4)^2$ / $f'(4.3) \approx -3 + 10(.3) - 6(.3)^2 = -.54$

c) Write the fourth-degree Taylor polynomial for $g(x) = \int_4^x f(t)dt$ about $x = 4$.

$g'(x) = f(x)$ / Integrating my original taylor series 1 step at a time

$\int_4^x 7dt = 7t\Big|_4^x = 7(x-4)$

$\int_4^x -3(t-4)dt = \frac{-3(t-4)^2}{2}\Big|_4^x = \frac{-3}{2}(x-4)^2 - \left(\frac{-3}{2}(4-4)^2\right) = \frac{-3}{2}(x-4)^2$

d) Can $f(3)$ be determined from the information given? Justify your answer.

$g(x) \approx 7(x-4) - \frac{3}{2}(x-4)^2 + \frac{5}{3}(x-4)^3 - \frac{2(x-4)^4}{4}$

d) No, b/c the series is centered at $x=4$ so I only know what's happening at 4, not 3.