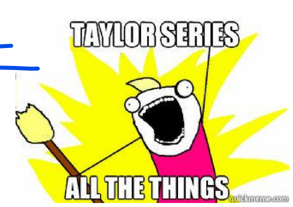
BC Calculus Taylor Series Notesheet **Name:** ____________________________

Taylor Polynomials can be used to approximate other elementary functions such as $y = \sin x$, $y = e^x$, and $y = \ln x$.



$f''(x) = -\sin x \rightarrow f''(0) = 0$

$f(0) = \sin 0 = 0$

Example 1 Find the equation of the tangent line for $f(x) = \sin x$ at $x = 0$, then use it to approximate $\sin(0.2)$. Is this an over or under approximation of $\sin(0.2)$?

$f'(x) = \cos x$

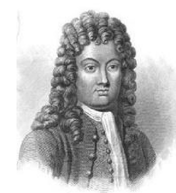
$f'(0) = \cos 0 = 1$

Tangent Line $y - 0 = 1(x - 0) \rightarrow y = x$

$\sin(0.2) \approx .2 \rightarrow$ overestimate

The equation of the tangent line used in example 1 is called a **first degree Taylor polynomial**. Taylor polynomials of higher degree can be used to obtain increasingly better approximations of non-polynomial functions with a certain **radius** from a **center of approximation** $x = c$.

Definition of an nth degree Taylor Polynomial



If f has n derivatives at $x = c$, then the polynomial

$$\boxed{P_n(x) = f(c) + f'(c)(x-c)} + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

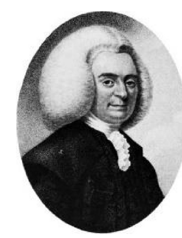
Linear Approx using a tangent

is called the nth degree Taylor polynomial for f centered at c.

Note 1: A first-degree Taylor polynomial is a tangent line to f at c.

Note 2: $\frac{f^{(n)}(c)}{n!}$ is the coefficient of the $(x-c)^n$ term.

Definition of nth degree Maclaurin Polynomial → Taylor series w/ $c = 0$



If the Taylor polynomial is centered at 0 ($c = 0$), then

$$P_n(x) = f(c) + f'(c)(x) + \frac{f''(c)}{2!}(x)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x)^n$$

is called the nth degree Maclaurin polynomial for f.

→ Taylor centered at $x = 0$

Example 3 Find the Maclaurin polynomial of degree $n = 7$ for $f(x) = \sin x$. Then use $P_7(x)$ to approximate the value of $\sin(0.1)$ using correct notation. Find the error for your approximation.

5, $f'(x) = \cos x \rightarrow$ at $0 \rightarrow f'(0) = 1$

6, $f''(x) = -\sin x \rightarrow$ " " $f''(0) = 0$

7, $f'''(x) = -\cos x \rightarrow f'''(0) = -1$

8, $f''''(x) = \sin x \rightarrow f^4(0) = 0$

$f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!}$

$f(0) \rightarrow 0$; $1(x-0) \rightarrow x$; $\frac{0(x-0)^2}{2!} \rightarrow 0$; $\frac{-1(x-0)^3}{3!} \rightarrow -\frac{x^3}{3!}$

$$P_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$P_7(.1) = .0998$

$\sin(.1) = .0998$ → error is really small

Example 4 List the first four non-zero terms and the nth term for the Maclaurin series for $f(x) = \sin x$.

↳ done :)

general term for $\sin x \rightarrow \frac{(-1)^n x^{2n+1}}{(2n+1)!} \rightarrow$ general term $(n \geq 0)$

$c = 0$ $\quad f(c) \to f(0) = \cos 0 = 1$

Example 5 Find the Maclaurin polynomial of degree $n = 7$ for $f(x) = \cos x$. Use it to find the first four non-zero terms and the nth term for the Maclaurin series for $f(x) = \cos x$.

$f'(x) = -\sin x \to 0$

$f''(x) = -\cos x \to$ at $c=0 \to -1$

$f^3(x) = \sin x \to 0$

$f^4(x) = \cos x \to$ at $0 \to 1$

$P_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$

$(-1)^n \frac{x^{2n}}{(2n)!} \to$ General Term

Example 6 Find the Maclaurin polynomial of degree $n = 4$ for $f(x) = e^x$. Use it to find the first ~~four~~ 5 non-zero terms and the nth term for the Maclaurin series for $f(x) = e^x$.

$f'(x) = e^x$ $\quad$ at $c=0$

$f''(x) = e^x$ $\quad e^0 = 1$

$f'''(x) = e^x$

$P_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

General term $\frac{x^n}{n!}$

Example 7 Find the Taylor polynomial of degree $n = 4$ for $f(x) = \ln x$ $\underbrace{\text{centered at } c = 1}_{\text{off center of } 0}$. Then use $P_4(x)$ to approximate the value of $\ln(1.1)$.

$f(x) = \ln x \to$ at $c=1 \to \ln 1 = 0$

$f'(x) = \frac{1}{x}$ or $x^{-1} \to 1$

$f''(x) = \frac{-1}{x^2} = -1x^{-2} \to -1$

$f'''(x) = \frac{2}{x^3} = 2x^{-3} \to 2$

$f^4(x) = \frac{-6}{x^4} = -6x^{-4} \to -6$

$0 + 1(x-1) - \frac{1(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} - \frac{6(x-1)^4}{4!}$

General $(n \geq 0)$ $\quad \frac{(-1)^n (x-1)^{n+1} n!}{(n+1)!}$

accuracy
$\ln(1.1) =$
$P_4(1.1) =$

Example 8 Suppose that g is a function which has continuous derivatives, and that $g(2) = 3$, $g'(2) = -4$, $g''(2) = 7$, and $g'''(2) = -5$. Write a Taylor polynomial of degree 3 for g centered at 2.

$3 - 4(x-2) + \frac{7(x-2)^2}{2!} - \frac{5(x-2)^3}{3!}$

$f(c)$

Example 9 Let $P(x) = 7 - 3(x-4) + 5(x-4)^2 - 2(x-4)^3 + 6(x-4)^4$ be the 4th degree Taylor polynomial for the function f about $x = 4$. Assume f has derivatives of all orders for all real numbers.

a) Find $f(4)$ and $f'''(4)$ → look at the coefficient of $(x-4)^3$

$f(4) = 7$

$-2 = \frac{f'''(4)}{3!} \to$ so $f'''(4) = -2 \cdot 3!$ or -12

b) Write the second-degree Taylor polynomial for f' about $x = 4$ and use it to approximate $f'(4.3)$.

$-3 + 10(x-4) - 6(x-4)^2$ $\qquad f'(4.3) \approx -3 + 10(4.3-4) - 6(4.3-4)^2$

c) Write the fourth-degree Taylor polynomial for $g(x) = \int_4^x f(t)dt$ about $x = 4$.

$\int_4^x 7\,dt = 7t\Big|_4^x = 7(x-4) \to$ keep the pattern going $\to 7(x-4) - \frac{3(x-4)^2}{2} + \frac{5(x-4)^3}{3} - \frac{2(x-4)^4}{4}$

d) Can $f(3)$ be determined from the information given? Justify your answer.

$f'(3)$ can't be determined → ① approx only
② info in this problem is about the function centered at 4