BC Calculus Taylor Series Notesheet B **Name:** ______________________ 1

There are 3 Maclaurin series that show up so often that it is a good idea to commit them to memory.

Maclaurin Series for $f(x) = e^x$

$$f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Maclaurin Series for $f(x) = \sin x$

$$f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

Maclaurin Series for $f(x) = \cos x$

$$f(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

If these series have been committed to memory, then they can be conveniently manipulated to find series representations of similar functions.

You can manipulate these series by using the following techniques:

Techniques for Manipulating Series

1. Substituting into a series for x
2. Multiply or divide the series by a constant and/or variable.
3. Add or subtract 2 series.
4. Differentiate or integrate a series, which may change the interval, but not the radius of convergence.
5. Recognize the series as the sum of a geometric power series.

Example 1 Find a Maclaurin series for $f(x) = \sin(x^2)$. Find the first four nonzero terms and the general term.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \quad + \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \quad + \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

Example 2 Find a Maclaurin series for $f(x) = x\cos(x)$. Find the first four nonzero terms and the general term.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \quad + \frac{(-1)^n x^{2n}}{(2n)!}$$

$$x\cos x = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \quad + \frac{(-1)^n x^{2n+1}}{(2n)!}$$

Example 3 Find a Maclaurin series for $f(x) = 4e^{x^2}$ Find the first four nonzero terms and the general term.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \quad + \frac{x^n}{n!}$$

$$4e^{x^2} = 4 + 4x^2 + \frac{4x^4}{2!} + \frac{4x^6}{3!} + \cdot + \frac{4x^{2n}}{n!}$$

Example 4 Find the first 3 nonzero terms in the Maclaurin series for $f(x) = e^x \sin x$.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + .$$

start w/ $1 \cdot x$ $\quad x\,x$ $\quad \frac{x^2}{2}\,x$

$1 \; \frac{-x^3}{6}$ $\quad x \; \frac{-x^3}{6}$

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} +$$

$1 \; \frac{x^5}{120}$

$$\boxed{x + x^2 - \frac{x^3}{6} + \frac{x^3}{2}}$$

combine Like terms $\rightarrow x + x^2 + \frac{x^3}{3}$

Series Free Response Examples

Example 5 The function g has derivatives of all orders, and the Maclaurin series for g is

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \cdots.$$

a) Using the ratio test, determine the interval of convergence of the Maclaurin series for g.

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+3}}{2n+5}\cdot\frac{2n+3}{(-1)^n x^{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{(-1)x^2(2n+3)}{(2n+5)}\right| \to |x^2|<1 \quad -1 \leq x \leq 1$$

check $x=-1 \to \frac{(-1)^{3n+1}}{2n+3} \to$ converges AST

check $x=1$ $\frac{(-1)^n}{2n+3}$ converges AST

b) The Maclaurin series for g evaluated at $x = \frac{1}{2}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $g\left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\frac{17}{120}$. Show that this approximation differs from $g\left(\frac{1}{2}\right)$ by less that $\frac{1}{200}$.

$$\left|g\left(\tfrac{1}{2}\right) - \frac{17}{120}\right| \leq \frac{\left(\frac{1}{2}\right)^5}{7} \to \frac{1}{32}\cdot\frac{1}{7} \to \frac{1}{224} < \frac{1}{200}$$

c) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$.

$$\frac{1}{3} - \frac{3x^2}{5} + \frac{5x^4}{7} \qquad \frac{(-1)^n(2n+1)x^{2n}}{2n+3}$$

Example 6 A function f has derivatives of all orders at $x = 0$. Let $P_n(x)$ denote the nth-degree Taylor polynomial for f about $x = 0$. (maclaurin)

a) It is known that $f(0) = -4$ and $P_1\left(\frac{1}{2}\right) = -3$. Show that $f'(0) = 2$.

$$P_1(x) = f(0) + f'(0)\,x$$

$$-3 = -4 + f'(0)\,\tfrac{1}{2}$$

1st add 4

$1 = f'(0)\,\frac{1}{2} \to$ multiply by 2

$2 = f'(0)$ ✓

b) It is known that $f''(0) = -\frac{2}{3}$ and $f'''(0) = \frac{1}{3}$. Find $P_3(x)$.

$$P_3(x) = f(0) + f'(0)\,x + \frac{f''(0)\,x^2}{2} + \frac{f'''(0)\,x^3}{6}$$

$$P_3(x) = -4 + 2x - \frac{\frac{2}{3}x^2}{2} + \frac{\frac{1}{3}x^3}{6}$$

c) The function h has first derivative given by $h'(x) = f(2x)$. It is known that $h(0) = 7$. Find the third-degree Taylor polynomial for h about $x = 0$.

maclaurin

Now $\int -4 + 4x - \frac{4x^2}{3} + \frac{8x^3}{18}$ (replacing x w/ 2x)

$$\boxed{7 - 4x + 2x^2 - \frac{4x^3}{9}}$$