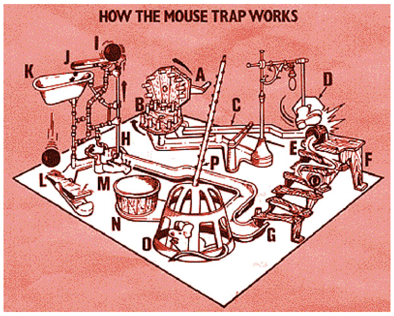
AB Calculus: The Chain Rule **Name:** ______________________________



In the board game mousetrap, players are tasked with building a contraption that will capture the other mice in the game. To win, a player starts a chain reaction by turning a crank that moves a boot that kicks a marble that goes down a chute causing another ball to fall out of a tub and hit a lever that flips a man into a bucket, causing the trap to come down. The Chain Rule actually gets its name because it is a similar chain reaction whereby one action triggers another, which triggers another, which triggers another.

In the image above, the turning of the crank at point A travels all the way through the board, eventually trapping the mouse at point P. If we wanted to know the change of P with respect to A, We would need to start by finding the rate of change of B, the next stage in the chain after A, and work our way all the way through to P.

$$\frac{dP}{dA}=\frac{dB}{dA}\cdot\frac{dC}{dB}\cdot\frac{dD}{dC}\cdot\frac{dE}{dD}\cdot\frac{dF}{dE}\cdot\frac{dG}{dF}\cdot\frac{dH}{dG}\cdot\frac{dI}{dH}\cdot\frac{dJ}{dI}\cdot\frac{dK}{dJ}\cdot\frac{dL}{dK}\cdot\frac{dM}{dL}\cdot\frac{dN}{dM}\cdot\frac{dO}{dN}\cdot\frac{dP}{dO}$$

The Chain Rule is our weapon for deriving composite functions, or functions (other than just plain x) within other functions. Here are some examples of the types of functions the chain rule will allow us to differentiate.

O outside I inside

Do Not Need Chain Rule	Need Chain Rule
$y = x^2 - 1$	$y = \sqrt{x^2 - 1}$
$y = \sin x$	$y = \sin 5x$
$y = 3x - 2$	$y = (x^2 - 1)^7$
$y = x - \tan x$	$y = x - \tan(x^2)$

O $\sqrt{x}$
I x^2-1

O $\sin x$
I $5x$

O x^7
I x^2-1

O $x - \tan x$
inside of $\tan x$ is x^2

The Chain Rule

$\frac{dy}{du}$ $\frac{du}{dx}$ $\frac{dy}{du}\,\frac{du}{dx}=\frac{dy}{dx}$

If $\boxed{y = f(u)}$ is a differentiable function of u and $\boxed{u = g(x)}$ is a differentiable function of x, then $y = f(g(x))$ is a differentiable function of x and

$y=f(g(x))$

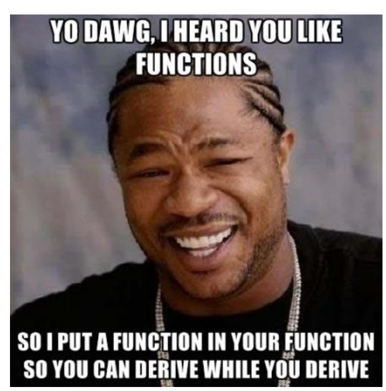
$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx} \text{ OR } \frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

$\frac{dy}{dx} = f'(g(x))\cdot g'(x)$

Composite functions have an "inside function" and an "outside function." Another way to look at this would be:

$$\frac{d}{dx}[f(g(x))] = f'(g(x))\,g'(x)$$

Derivative of the "outside function" … leave the "inside function" alone. × Derivative of the "inside function"



The toughest part at first is identifying the "inside" and the "outside" functions.

Example 1: Identify the inside and outside function in each of the following composite functions.

a) $f(x) = \cos(5x^2)$

O $\cos t$

I $5x^2$

b) $g(x) = \sqrt{3x+1}$

O $\sqrt{t}$

I $3x+1$

c) $f(x) = \sin^2 x \rightarrow (\sin x)^2$

O t^2

I $\sin x$

d) $f(x) = \dfrac{1}{5x+1}$

O $\dfrac{1}{t}$

I $5x+1$

Example 2: Find the derivative of each of the following functions.

a) $f(x) = \cos(5x^2)$

Functions Derivatives

O $\cos t \rightarrow -\sin t$

I $5x^2 \rightarrow 10x$

$$\boxed{f'(x) = -\sin(5x^2) \cdot 10x}$$

b) $g(x) = \sqrt{3x+1}$

O $\sqrt{t} \rightarrow \dfrac{1}{2\sqrt{t}}$

I $3x+1 \rightarrow 3$

$$\boxed{g'(x) = \frac{1}{2\sqrt{3x+1}} \cdot 3}$$

c) $f(x) = \sin^2 x$

O: $t^2 \rightarrow 2t$

I: $\sin x \rightarrow \cos x$

$$\boxed{f'(x) = 2(\sin x)\cos x}$$

d) $f(x) = \dfrac{1}{5x+1}$

algebra derivative

O $\dfrac{1}{t} \rightarrow 1t^{-1} \rightarrow -1t^{-2}$

I $5x+1 \rightarrow 5$

$$\boxed{f'(x) = -1(5x+1)^{-2}\,5} \text{ or } \boxed{\frac{-5}{(5x+1)^2}}$$

Deriving and identifying the inside and outside functions gets tougher when you have functions with multiple inside functions. When this happens, you peel the function like an onion, taking the derivative of the outside of the remaining function that has not been derived yet and keeping the inside functions the same until you work your way all the way to the innermost function.

Example 3: Identify the functions that make up the composite functions below from inside to outside.

a) $f(x) = \sin^3\sqrt{4x+1}$

O $x^3 \rightarrow 3x^2$

$\sin x \rightarrow \cos x$

$\sqrt{x} \rightarrow \dfrac{1}{2\sqrt{x}}$

I. $4x+1 \rightarrow 4$

b) $g(x) = \dfrac{3}{\sqrt{\tan(4x^2+1)}}$

algebra derivative

O. $\dfrac{3}{x} \rightarrow 3x^{-1} \rightarrow -3x^{-2}$

algebra

$\sqrt{x} \rightarrow x^{\frac{1}{2}} \rightarrow \frac{1}{2}x^{-\frac{1}{2}}$

$\tan x \rightarrow \sec^2 x$

I. $4x^2+1 \rightarrow 8x$

Example 4: Find the derivative of each of the following

a) $f(x) = \sin^3 \sqrt{4x+1}$

$$f'(x) = 3\left(\sin\sqrt{4x+1}\right)^2 \cdot \cos\sqrt{4x+1} \cdot \frac{1}{2\sqrt{4x+1}} \cdot 4$$

b) $g(x) = \dfrac{3}{\sqrt{\tan(4x^2+1)}}$

$$g'(x) = -3\left(\sqrt{\tan(4x^2+1)}\right)^{-2} \cdot \frac{1}{2}\left(\tan(4x^2+1)\right)^{-\frac{1}{2}} \cdot \sec^2(4x^2+1) \cdot 8x$$

Combining All the Rules

Example 5: Find the derivative of each of the following

a) $f(x) = (x^2+1)\sqrt{2x-3}$ $\quad f(x) = (x^2+1)(2x-3)^{\frac{1}{2}}$

$$f'(x) = (x^2+1) \cdot \frac{1}{2}(2x-3)^{-\frac{1}{2}} \cdot 2 + (2x) \cdot (2x-3)^{\frac{1}{2}}$$

b) $g(t) = \left(\dfrac{t-2}{2t+1}\right)^9$

$2t+1-2t+4$

$$g'(t) = 9\left(\frac{t-2}{2t+1}\right)^8 \cdot \left(\frac{(2t+1)(1)-(t-2)(2)}{(2t+1)^2}\right) \xrightarrow{\text{Practice}} 9\left(\frac{t-2}{2t+1}\right)^8\left(\frac{5}{(2t+1)^2}\right) \to \boxed{\frac{45(t-2)^8}{(2t+1)^{10}}}$$

Example 6: For each of the following, use the fact that $g(5) = -3$, $g'(5) = 6$, $h(5) = 3$, and $h'(5) = -2$ to find $f'(5)$, if possible. If it is not possible, state what additional information is needed to find the value.

a) $f(x) = g(x)h(x)$

$$f'(x) = g(x)\,h'(x) + g'(x)\,h(x) \to f'(5) = g(5)\,h'(5) + g'(5)\cdot h(5)$$

$$(-3)(-2) + (6)(3) = \boxed{24}$$

b) $f(x) = g(h(x))$

$$f'(x) = g'(h(x))\,h'(x) \to f'(5) = g'(h(5))\cdot h'(5) \to g'(3)\,(-2) \to g'(3) \text{ is } \underline{\text{unknown}}$$

c) $f(x) = \dfrac{g(x)}{h(x)}$

$$f'(x) = \frac{h(x)\,g'(x) - g(x)\,h'(x)}{[h(x)]^2} \Rightarrow f'(5) = \frac{(3)(6)-(-3)(-2)}{3^2} = \frac{18-6}{9} = \frac{12}{9} \boxed{= \frac{4}{3}}$$

d) $f(x) = [g(x)]^3$

$$f'(x) = 3[g(x)]^2\,g'(x) \to f'(5) = 3[g(5)]^2\,g'(5) = 3(-3)^2\,6$$

$$= 3(9)(6) = \boxed{162}$$

Note: you may see the expression $[g(x)]^3$ as $g^3(x)$.